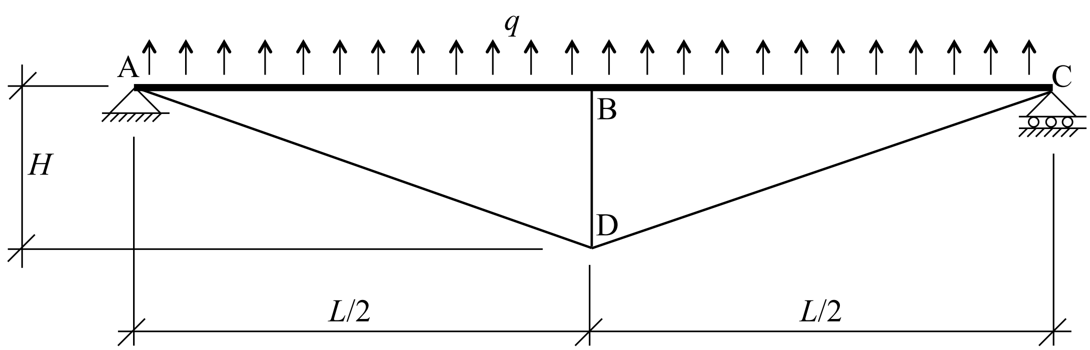
Beam with Pre-tensioned Truss Under

The beam shown below is supported by the three truss members AD, DC, and BD, and this structure is analyzed for wind uplift and pre-tensioning in the members AD and DC. The objective is to determine the amount of pre-tensioning that will prevent compression in those two members when the wind uplift acts. This is a reasonable design requirement because any sudden shift in axial force in members AD and DC from compression to tension would be unsafe. However, the pre-tensioning will cause point B to displace upwards, and this displacement must be monitored.



Input values in kN and m

```
q = 2;
L = 6;
H = 1;
```

The length of truss members AD and CD is:

$$LAD = \sqrt{H^2 + \left(\frac{L}{2}\right)^2} \ // \ N$$

which yields: 3.16228

The beam and the vertical member BD are made of rectangular cross-sections made of wood, while the inclined rods have circular steel cross-sections:

$$\text{Ewood} = 9\,500\,000;$$
$$\text{Esteel} = 200\,000\,000;$$
$$\text{rrods} = 0.01;$$
$$\text{bbeam} = 0.076;$$
$$\text{hbeam} = 0.235;$$
$$\text{bvertical} = 0.076;$$
$$\text{hvertical} = 0.089;$$

That gives the following cross-sectional constants:

$$\text{EIbeam} = \text{Ewood}\,\frac{\text{bbeam hbeam}^3}{12}$$

which yields: 780.835

$$\text{EAbeam} = \text{Ewood bbeam hbeam}$$

which yields: 169 670.

$$\text{EArods} = \text{Esteel}\,\pi\,\text{rrods}^2$$

which yields: 62 831.9

$$\text{EAvertical} = \text{Ewood bvertical hvertical}$$

which yields: 64 258.

Degree of static indeterminacy

The first step in the analysis of any truss or frame structure is to calculate the degree of static indeterminacy, DSI. Here we use a formula that employs these symbols:

f = number of unknown **f**orces in each member
m = number of **m**embers
r = number of support **r**eactions
e = number of equilibrium **e**quations at each joint
j = number of **j**oints
h = number of **h**inges, i.e., releases of internal forces

This structure has both truss and frame members; hence, the counting can be done in two different ways. First, we count all members as beams, and introduce hinges in the truss members to ensure that they do not take any bending moment:

```
f = 3;
m = 5;
r = 3;
e = 3;
j = 4;
h = 5;
```

The degree of static indeterminacy is:

```
DSI = (f m + r) - (e j + h)
```

which yields: 1

The alternative approach is to explicitly differentiate between the number of unknown forces in a truss member and a beam member, and also the different number of equations that can be established for a truss joint and a beam joint. It is now unnecessary to introduce any artifical hinges:

```
fbeam = 3;
ftruss = 1;
mbeam = 2;
mtruss = 3;
r = 3;
ebeam = 3;
etruss = 2;
jbeam = 3;
jtruss = 1;
h = 0;
```

The degree of static indeterminacy remains the same:

```
DSI = (fbeam mbeam + ftruss mtruss + r) - (ebeam jbeam + etruss jtruss + h)
```

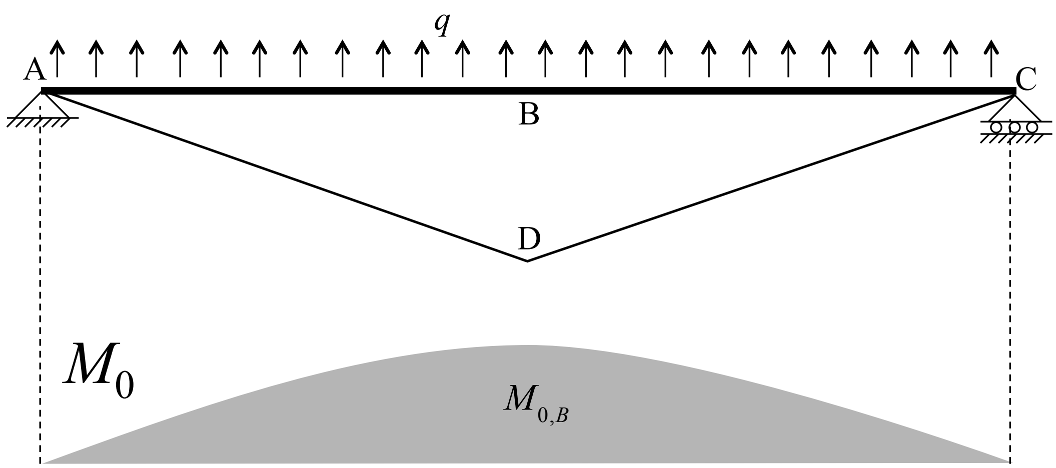
which yields: 1

Flexibility method to determine internal forces due to wind

The flexibility method is selected for the analysis, and the axial force in BD is selected as the redundant force. This means we must analyze the statically determinate structure in which member BD is "cut."

Force due to wind on determinate structure

First we address the wind load, noticing that distributed load on inclined surfaces can be decomposed to act on projected surfaces, and that we split the bending moment diagram to facilitate later use of quick integration formulas in the virtual work method:

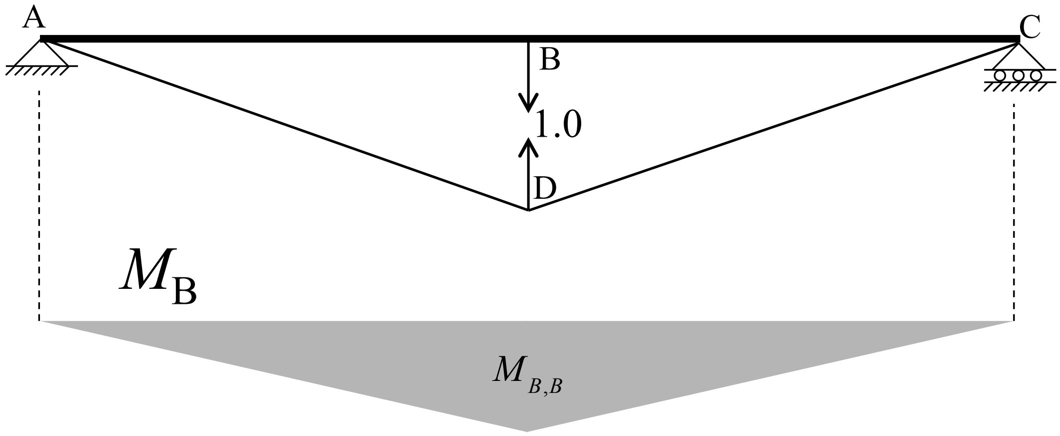


$$M0B = \frac{q\,L^2}{8}$$

which yields: 9

Force due to unit load on determinate structure

The unit force along the redundant (tension positive) creates both axial forces and bending moment. The load at B creates only bending moment. Importantly, the load at D creates both axial forces AND bending moment. It is the inclusion of that bending moment that accounts for the extra “gap” from horizontal displacement at C.



First addressing the unit load at B, the beam is essentialy a simply supported beam with length *L:*

$$MBB = \frac{L}{4} \text{ // N}$$

which yields: 1.5

Axial force in member DB is:

$$NBDB = 1;$$

Axial force in members AD and CD by equilibrium at joint D:

$$NBAD = \frac{NBDB}{2} \frac{LAD}{H} \text{ // N}$$

which yields: 1.58114

Axial force in beam by equilibrium at joint A:

$$NBAC = NBAD \frac{\frac{L}{2}}{LAD} \text{ // N}$$

which yields: 1.5

Compatibility equation

We need "gap opening" due to the wind load, ΔB0, and the "gap closing" due to the unit load, ΔBB. Notice how the quick integration formulas can be used although the beam is curved, when we use the actual beam length determined earlier. First combine M0 with MB:

$$\triangle B0 = -\left(\frac{5\ MBB\ M0B}{12\ EIbeam}\ L\right)$$

which yields: -0.0432229

Next combine the internal forces due to the unit load, with themselves, making sure the units of all the internal forces are "internal force per kN of the unit load" and noticing the transformation in the integrand of the integral in polar coordinates:

$$\triangle BB = \frac{MBB^2}{3\ EIbeam}\ L + \frac{NBAC^2}{EAbeam}\ L + 2\ \frac{NBAD^2}{EArods}\ LAD + \frac{NBDB^2}{EAvertical}\ H$$

which yields: 0.00610983

Solving the compatibility equation yields the axial force in member BD, with positive being tension:

$$XB = -\frac{\triangle B0}{\triangle BB}$$

which yields: 7.07432

Final axial force in AD and CD due to wind

The final bending moment diagram is $M_{\text{total}} = M_0 + M_B\ X_B$ but of greater interest here is the axial force in the truss members AD and CD, which we soon want to cancel out by pre-tensioning. That axial force due to wind is, after making sure the unit of $N_{B,\text{AD}}$ is "kN per kN of force along the redundant:"

NADwind = NBAD XB

which yields: 11.1855

Flexibility method to find internal forces due to pre-tensioning

Using the flexibility method we seek the "gap opening" ΔB0 due to pre-tensioning. This displacement is calculated using virtual work (minus sign because pre-tensioning is considered to be a shortening, a phenomenon associated with compression):

△B0pretensioning = -2 NBAD pretensioning;

Solving the compatibility equation yields the axial force in member BD:

```
XBpretensioning = - △B0pretensioning / △BB;
```

That means the bending moment in the beam is (positive here means tension at the bottom)

```
MBpretensioning = MBB XBpretensioning;
```

The axial force in AD and CD is:

```
NADpretensioning = NBAD XBpretensioning;
```

The axial force in the beam AC is:

```
NACpretensioning = NBAC XBpretensioning;
```

Required pre-tensioning to avoid compression in AD and CD

Set the axial force in member AD due to wind equal to *minus* the axial force due to pre-tensioning, and solve for the length change (pre-tensioning) needed to cancel out the axial force (notice the result can become unreasonably large):

```
solution = Solve[NADwind == -NADpretensioning, pretensioning]
```

which yields: `{{pretensioning → -0.0136683}}`

That means the following shortening of AD and CD in *millimeter*, mm:

```
1000 pretensioning /. solution[[1]]
```

which yields: `-13.6683`

That pre-tensioning causes the following internal forces in the structure:

```
XBpretensioning /. solution[[1]]
```

which yields: `-7.07432`

```
momentAtBdueToPretensioning = MBpretensioning /. solution[[1]]
```

which yields: `-10.6115`

```
NADpretensioning /. solution[[1]]
```

which yields: -11.1855

```
NACpretensioning /. solution[[1]]
```

which yields: -10.6115

Uplift at mid-span due to pre-tensioning

When members AD and CD are shortened (pre-tensioned) then point B displaces upward. It is discussed below why the virtual work method is not suitable to find this displacement. However, a straightforward alternative is the moment-area method. The tangential deviation at A, namely t_{AB}=moment of M/EI about A, is the sought displacement (uplift), here presented in *millimeter* (mm):

$$\left(\frac{1}{2}\,\frac{\text{Abs}[\text{momentAtBdueToPretensioning}]}{\text{EIbeam}}\,\frac{\text{L}}{2}\right)\frac{2\,\text{L}}{3}\,1000$$

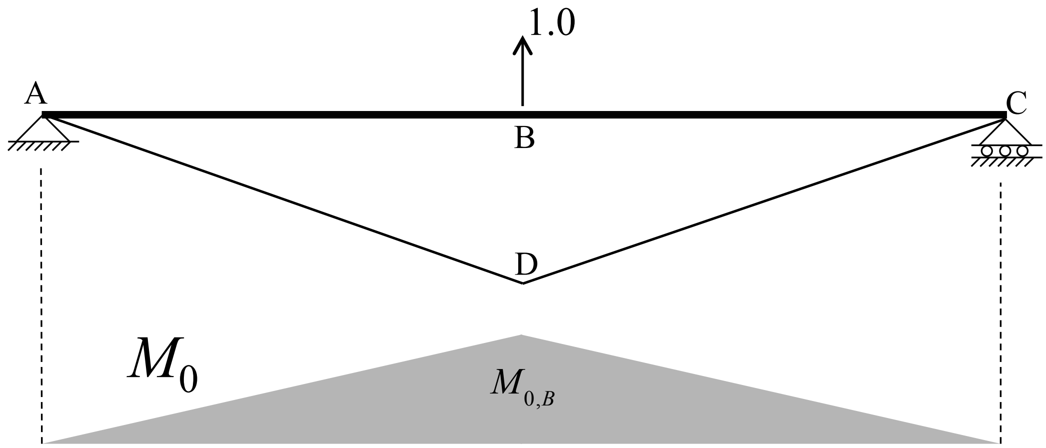
which yields: 81.5395

Why virtual work does not determine the displacement due to pre-tensioning

Normally the virtual work method is ideal for determining displacements and rotations. We normally place a unit load or moment at the location where we want to determine the displacement or rotation. However, the deformations from the pre-tensioning addressed in this document do not come from an externally applied force. The forces due to pre-tensioning are self-equilibrating. Below it is shown how those forces do not cause net virtual work. Another strike against the virtual work method in this case is that the structure is statically indeterminate. That means we have to carry out the flexibility method all over again just to find the internal forces due to a unit load applied at the location we want the displacement:

Flexibility method to find internal forces due to unit load at B

The unit force on the statically determinate structure analyzed in the flexibility method:



Internal forces in the determinate “0 system:”

$$\text{M0BunitLoadAtB} = \frac{L}{4} \; // \; N$$

which yields: 1.5

Displacement needed in the compatibility equation:

$$\triangle\text{B0unitLoadAtB} = -\frac{\text{MBB M0BunitLoadAtB}}{3\ \text{EIbeam}} L$$

which yields: -0.00576306

Solving the compatibility equation:

$$\text{XBunitLoadAtB} = -\frac{\triangle\text{B0unitLoadAtB}}{\triangle\text{BB}}$$

which yields: 0.943243

The associated axial force in AD:

$$\text{NADunitLoadAtB} = \text{NBAD XBunitLoadAtB}$$

which yields: 1.4914

The associated axial force in the beam AC:

```
NACunitLoadAtB = NBAC XBunitLoadAtB
```

which yields: 1.41486

And the associated bending moment at B, using $M_{\text{total}} = M_0 + M_B\, X_B$, counting tension at the top as positive here:

```
MBBunitLoadAtB = M0BunitLoadAtB - MBB XBunitLoadAtB
```

which yields: 0.0851352

Virtual work to find uplift at B due to pre-tensioning

In this application of the unit virtual load method we combine the internal forces due to pre-tensioning and unit load at B. These are facts about the signs:

Pre-tensioning causes TENSION AT THE TOP of the beam
Unit load at B causes TENSION AT THE TOP of the beam

Pre-tensioning causes COMPRESSION in the beam
Unit load at B causes TENSION in the beam

Pre-tensioning causes COMPRESSION in the vertical member BD
Unit load at B causes TENSION in the vertical member BD

Pre-tensioning causes TENSION in members AD and CD
Unit load at B causes COMPRESSION in members AD and CD

Hence the following use of signs and absolute values, leading to to the zero virtual work conclusion for self-equilibrating forces:

$$\triangle B = \frac{\text{Abs[MBBunitLoadAtB] Abs[momentAtBdueToPretensioning]}}{\text{3 EIbeam}} L - \frac{\text{Abs[NACunitLoadAtB] Abs[NACpretensioning /. solution[[1]]]}}{\text{EAbeam}} L - \frac{\text{Abs[XBunitLoadAtB] Abs[XBpretensioning /. solution[[1]]]}}{\text{EAvertical}} H - 2\,\frac{\text{Abs[NADunitLoadAtB] Abs[NADpretensioning /. solution[[1]]]}}{\text{EArods}} \text{LAD}$$

which yields: 3.46945×10^{-18}