Poisson Point Processes

The popular Poisson process is a Bernoulli sequence in which trials are carried out at every time instant. I.e., the time between trials is zero and the number of trials is infinite. From here on, a trial that yields success will be called "occurrence." Under the assumptions that 1) an occurrence is equally likely to occur at any time instant, 2) any occurrence is independent of what happened before, and 3) only one occurrence can happen at a particular time, the number of successes, x, in a time interval, T, is given by the Poisson distribution:

$$p(x) = \frac{(\lambda \cdot T)^x}{x!} e^{-\lambda \cdot T} \tag{1}$$

where λ is the rate of occurrences, i.e., the mean number of occurrences per unit time. Provided this basis, the Poisson process is referred to as a counting process; it counts occurrences in specific time intervals. The time between occurrences, t, has the exponential distribution:

$$f(t) = \lambda \cdot e^{-\lambda \cdot t} \tag{2}$$

The mean time between occurrences is $1/\lambda$ and is habitually called the "return period." Notice that realizations of a Poisson process is easily generated by generating outcomes of t, i.e., the random time between occurrences. It is also noted that the Poisson process has only one parameter, i.e., λ. However, there are different ways of expressing this rate. For example, expressions like "2% in 50" appear in earthquake engineering as proxies for λ. To understand their meaning, consider the probability of any non-zero number of occurrences during a time interval T, which is provided by Eq. (1):

$$p(1) + p(2) + \cdots = 1 - p(0) = 1 - e^{-\lambda \cdot T} \tag{3}$$

From Eq. (3) one can solve for the rate that yields a 2% probability of occurrence in a 50-year time interval. Expressions like "1-in-50" are also encountered in building codes. This is a direct expression of the rate, i.e., λ=1/50 and consequentially the return period is 50 years. Table 1 exemplifies that the rate is not equal to the annual probability.

Table 1: Return periods, rates, and annual probabilities.

Return period, in years	Rate, i.e., mean annual frequency	Annual probability of occurrence
1	1	1/1.582
5	1/5	1/5.517
10	1/10	1/10.508
50	1/50	1/50.502
100	1/100	1/100.501
500	1/500	1/500.500
1,000	1/1,000	1/1000.500
10,000	1/10,000	1/10,000.500

Bayesian Inference

There are several ways to estimate the occurrence rate λ of a Poisson process. The simplest and least precise approach is to divide the number of observations in a time interval by the length of the interval. Another approach is to explore a fit between the probability distribution in Eq. (2) and the observed values of time between occurrences. A good fit would indicate that the exponential distribution is an appropriate probability distribution, and thus that the underlying model is indeed a Poisson process.

Another option is Bayesian updating. In accordance with the general Bayesian principle, the distribution parameter λ is then considered a random variable. A conjugate prior for λ in Eq. (1) is the gamma distribution:

$$f(\lambda) = \frac{\nu(\nu\lambda)^{k-1}}{\Gamma(k)} \exp(-\nu\lambda) \tag{4}$$

Conjugate priors retain their distribution type as a posterior, and the parameters of the distributions are in this case updated by the formulas

$$\begin{aligned} k'' &= k' + x \\ \nu'' &= \nu' + t \end{aligned} \tag{5}$$

where double-prime and prime identifies posterior and prior parameters, respectively, while x is the number of observed occurrences in t.