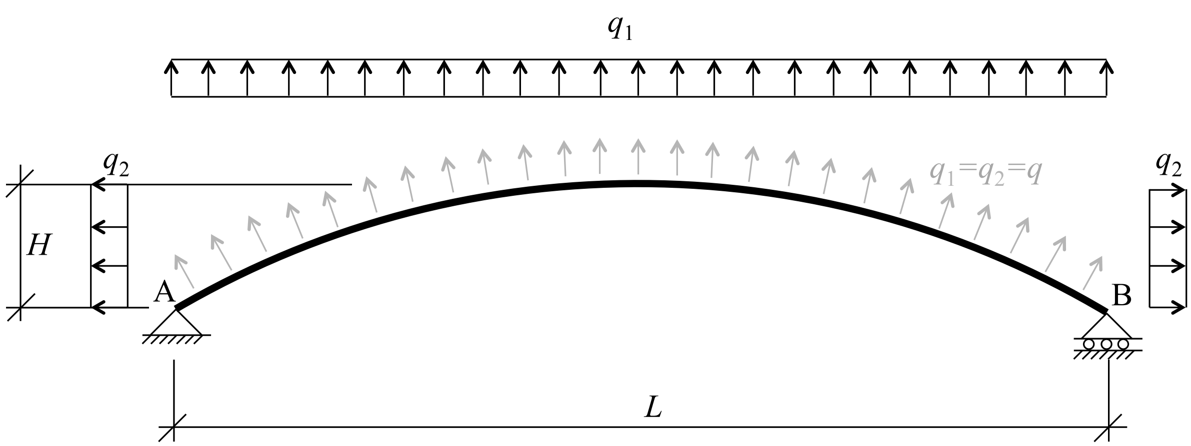
Curved Beam

The starting point for this analysis a statically determinate beam with a circular shape, as shown below. At first, the objective is to calculate internal forces and deformations due to q_1 and q_2. Notice that $q_1 = q_2$ implies uniform load along the beam as shown. Thereafter, the support at B is fixed and the buckling load of the resulting arch is determined. In Timoshenko's Strength of Materials, Part 2, we find formulas for beams that have significant cross-section height compared with the radius of initial curvature, but here thin beams, i.e., small height compared with radius, is addressed. This implies that the neutral axis is located at the centroid of the cross-section.



Input values

The beam has a circular steel cross-section with radius *r*:

```
E = 200 000 N/mm^2 ;
r = 100 mm ;
q1 = 2 kN/m ;
q2 = 2 kN/m ;
L = 10 m ;
H = 1.1 m ;
```

That means the cross-section area is:

```
A = π r^2 // N
```

which yields: 31 415.9 mm^2

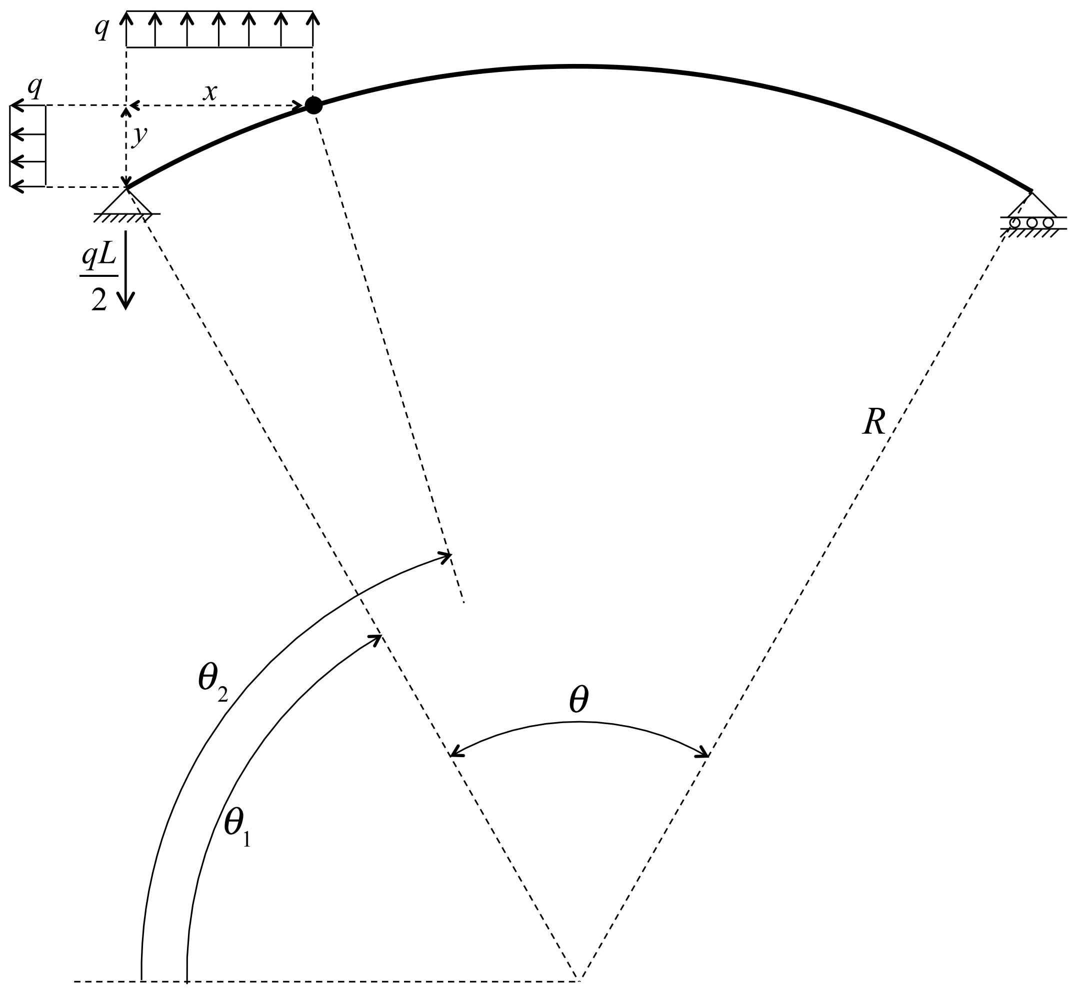
And the moment of inertia is:

```
I = π/4 r^4 // N
```

which yields: 7.85398×10^7 mm^4

Geometry

The following figure shows notation used when the beam is a circular segment. Notice that distributed load on an inclined line can be decomposed onto the horizontal and vertical projection of the line:



The radius that follows from the given geometry parameters is:

$$R = \frac{H}{2} + \frac{L^2}{8\,H}$$

which yields: 11.9136 m

The angle spanned by the circular segment from A to B is, in radians:

$$\theta = 2\,\text{ArcSin}\left[\frac{L}{2\,R}\right]$$

which yields: 0.866201

That result in degrees is:

$$\frac{\theta}{\pi}\,180$$

which yields: 49.6297

That means the actual length of the beam is:

$$\text{Lbeam} = \theta\,R$$

which yields: 10.3196 m

The angle of the beam's cross-section at the support is:

$$\theta 1 = \frac{\pi}{2} - \frac{\theta}{2}$$

which yields: 1.1377

That result in degrees is:

$$\frac{\theta 1}{\pi}\,180$$

which yields: 65.1852

Axial force

The downward support reaction at A is:

```
reaction = q1 L / 2
```

which yields: 10 kN

That reaction means the axial force at the support (in tension) is, by decomposition:

```
Nsupport = reaction Cos[θ1]
```

which yields: 4.19687 kN

Now consider a generic point a horizontal distance x from A, shown as a small solid circle in the figure above. At this point a "cut" is made perpendicular to the neutral axis, and the forces on that cross-section is found by equilibrium:

```
upwardForce = reaction - q1 x;
rightwardForce = q2 y;
```

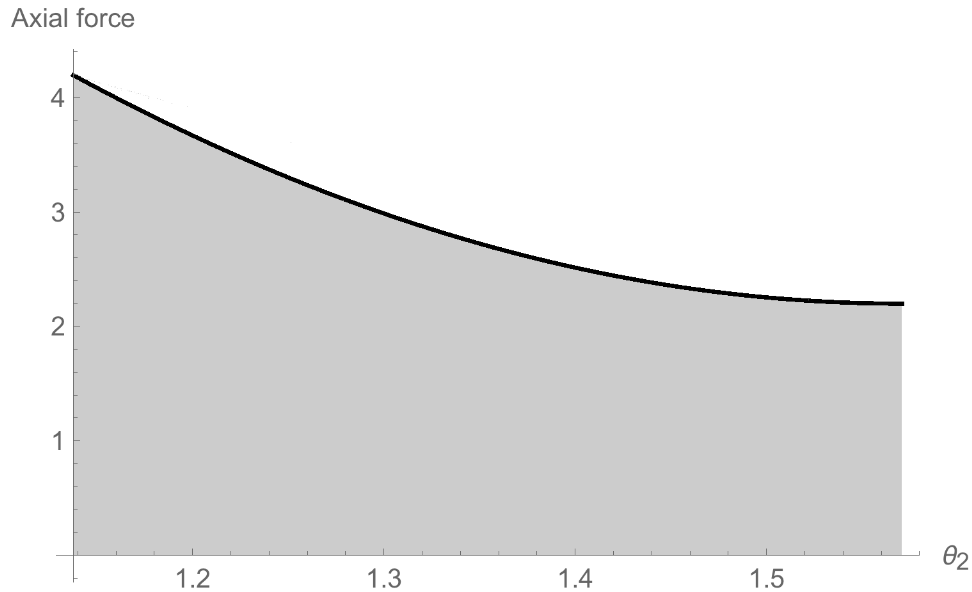
We seek expressions that are function of the angle θ_2 instead of x and y. Geometrical considerations yield:

```
x = R (Cos[θ1] - Cos[θ2]);
y = R (Sin[θ2] - Sin[θ1]);
```

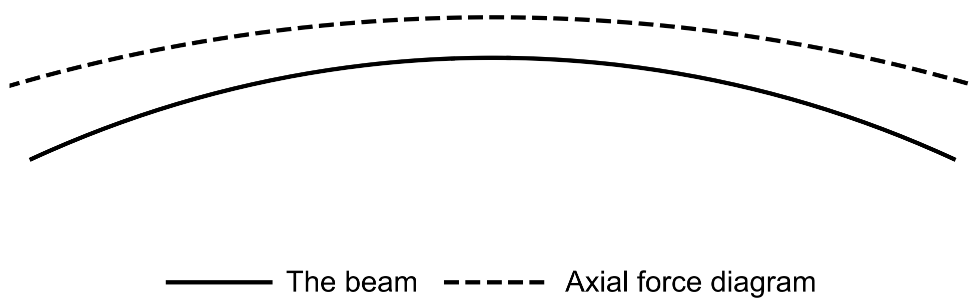
The upward and rightward forces on the cross-section are then decomposed into the axial direction and summed to give the axial force in a cross-section a horizontal distance x away from A, as a function of θ_2, and positive in tension:

```
axialForce = rightwardForce Sin[θ2] + upwardForce Cos[θ2];
```

That axial force can be plotted as a function of θ_2 (tension always positive):



That axial force can also be visualized in a polar plot:

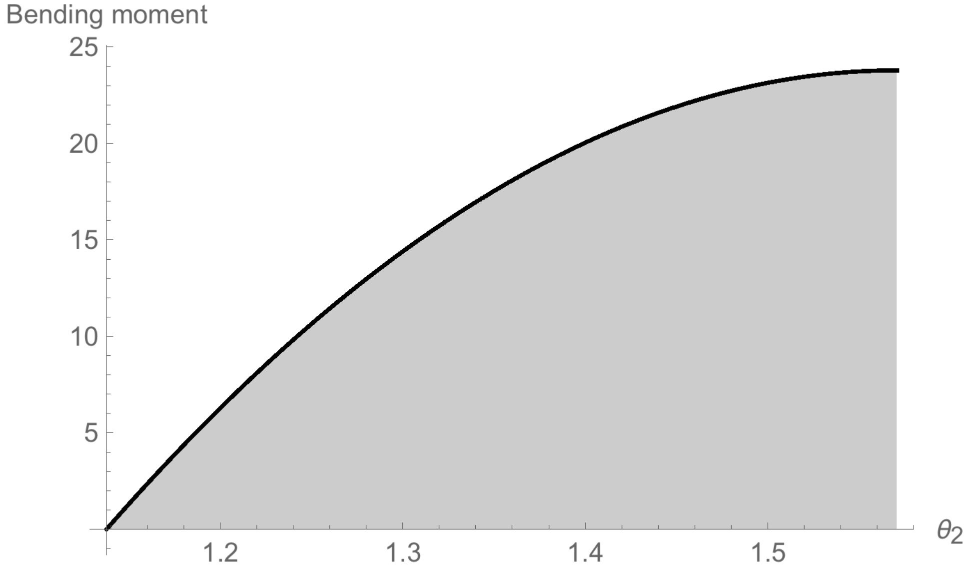


Bending moment

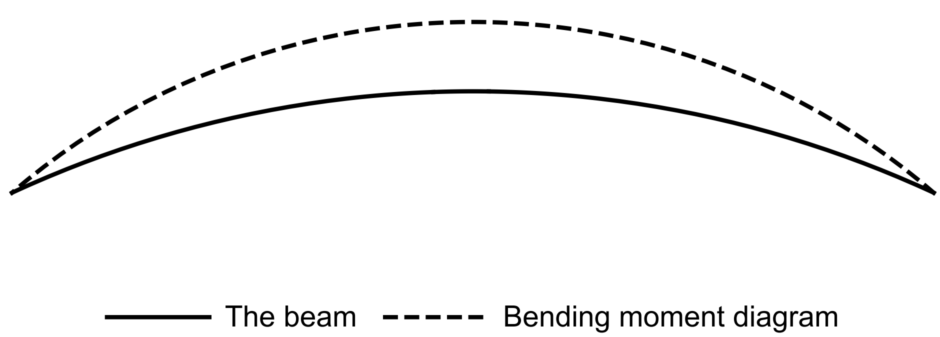
The approach followed above for the axial force can be applied to the bending moment too, namely calculating its value at the point located a distance *x* from A (tension at the top is considered positive):

```
bendingMoment = reaction x - q1 x^2/2 - q2 y^2/2;
```

That bending moment is negative for all θ_2-values, meaning there is tension at the top everywhere:

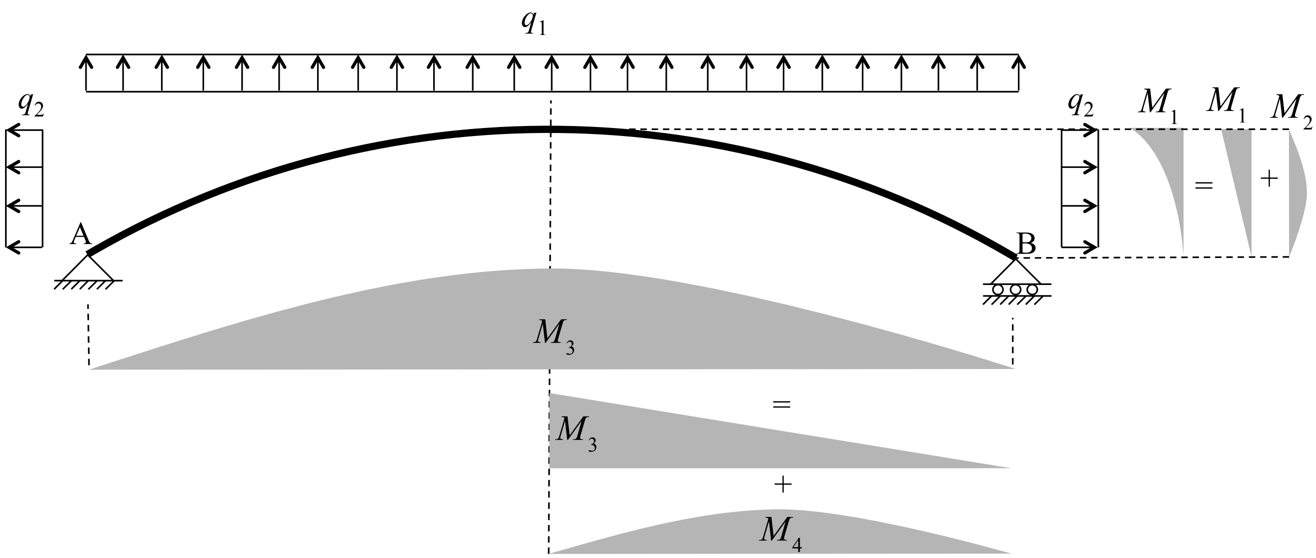


The bending moment can also be visualized in a polar plot, here appearing on the tension side:



Alternative approach to bending moment calculations

The considered beam has a pin support on the left, and a roller on the right, and is therefore a statically determinate simply supported beam. As above, we also observe that distributed load on inclined surfaces can be decomposed onto projected surfaces, as shown in the figure below. That means the bending moment diagram can be split into basic shapes (triangles and regular parabolas) to facilitate the use of quick integration formulas in the virtual work method.



The beam from midspan to B is essentially a cantilever with length *H* when seen as a horizontal projection:

$$M1 = q2 \frac{H^2}{2}$$

which yields: 1.21 m kN

$$M2 = \frac{q2\ H^2}{8} \ // \ N$$

which yields: 0.3025 m kN

The whole beam from A to B is essentially a simply supported beam when seen as a vertical projection:

$$M3 = \frac{q1\ L^2}{8}$$

which yields: 25 m kN

$$M4 = \frac{q1 \left(\frac{L}{2}\right)^2}{8} \ // \ N$$

which yields: 6.25 m kN

We can now compare bending moment values with the approach used earlier. Here is the favourable comparison at midspan between A and B:

```
bendingMoment /. θ2 → π/2
```

which yields: 23.79 m kN

```
M3 - M1
```

which yields: 23.79 m kN

A comparison of bending moment values at the midpoint between A and the top of the arc, namely a comparison at the point where $x=L/4$, is harder because at this location we do NOT have that y=H/2. For that reason, only a ballpark figure is obtained:

```
bendingMoment /. θ2 → ArcCos[(R Cos[θ1] - L/4)/R]
```

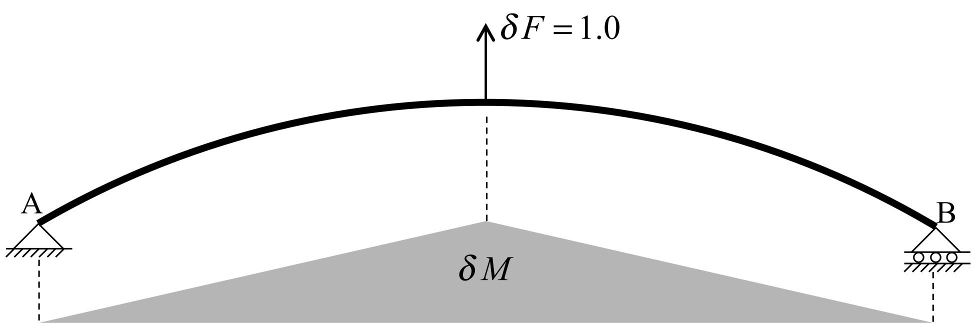
which yields: 18.0532 m kN

```
M2 + 1/2 M3 + M4 - 1/2 M1
```

which yields: 18.4475 m kN

Displacement at midspan

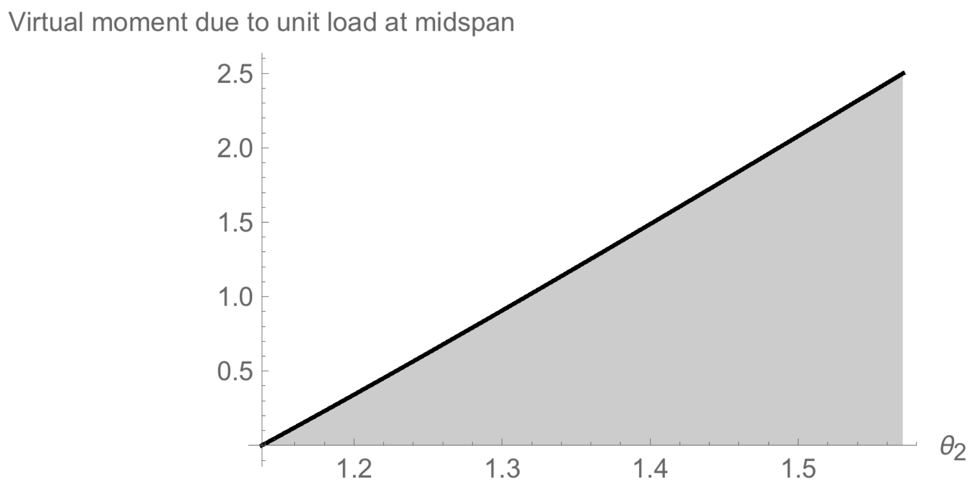
To determine the vertical displacement at midspan by the unit virtual load method we place a unit load at the location where we want the displacement, yielding the following bending moment diagram:




```
δF = 1 kN;
```

To carry out the integration of $\left(\delta \mathrm{M}\ \frac{M}{\mathrm{EI}}\right)$it is best to use the formulas for the moment diagrams expressed as function of θ_2 (tension at the top is considered positive):

$$\text{virtualMoment} = \frac{\delta F}{2} x;$$



The integration to obtain displacement due to flexure contains a factor 2 because there are 2 parts of the beam:

$$\triangle\text{midspanFlexure} = \frac{2}{\delta F} \int_{\theta 1}^{\frac{\pi}{2}} \left(\text{virtualMoment} \frac{\text{bendingMoment}}{\text{E I}} \right) \text{R}\, d\theta 2;$$

UnitConvert[△midspanFlexure, "mm"]

which yields: 16.0429 mm

Is it possible to use quick integration formulas? In other words, is the virtual bending moment diagram linear both along the *x* and *y* axes? Not exactly:

$$\delta M = \frac{\delta F\, L}{4};$$

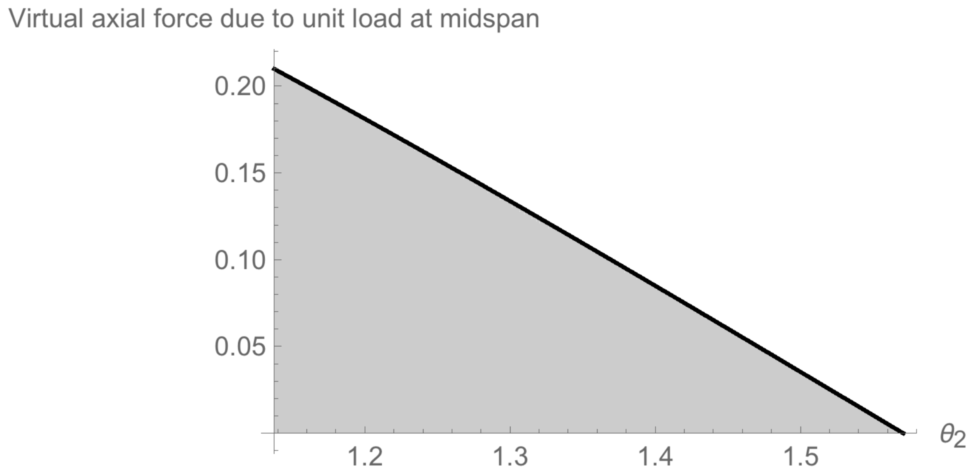
$$\triangle\text{midspanQuickIntegration} = \frac{1}{\delta F} \left(-\frac{2\, \delta M\, M1}{3\, E\, I} H + \frac{2\, \delta M\, M2}{3\, E\, I} H + \frac{\delta M\, M3}{3\, E\, I} L + \frac{\delta M\, M4}{3\, E\, I} L \right);$$

UnitConvert[△midspanQuickIntegration, "mm"]

which yields: 16.4727 mm

To include axial deformations we first determine the virtual axial force, using the same approach as earlier, with positive implying tension, and resulting in zero axial force at the apex:

$$\text{virtualAxialForce} = \frac{\delta F}{2} \text{Cos}[\theta 2];$$



The contribution to the midspan displacement from axial deformation is small, for the 2 parts of the beam, one on each side of the apex:

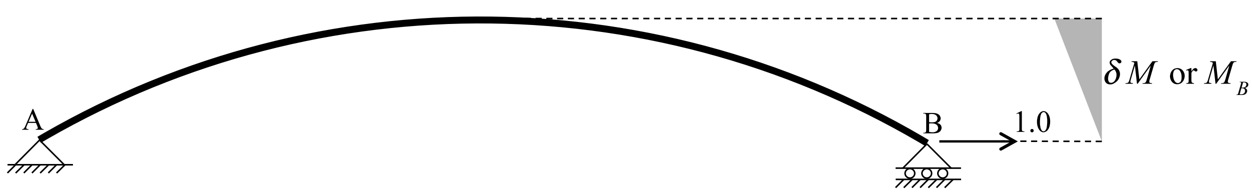
$$\triangle\text{midspanAxial} = \frac{2}{\delta F}\int_{\theta 1}^{\frac{\pi}{2}}\left(\text{virtualAxialForce}\ \frac{\text{axialForce}}{E\,A}\right) R\, d\theta 2;$$

```
UnitConvert[△midspanAxial, "mm"]
```

which yields: 0.000559952 mm

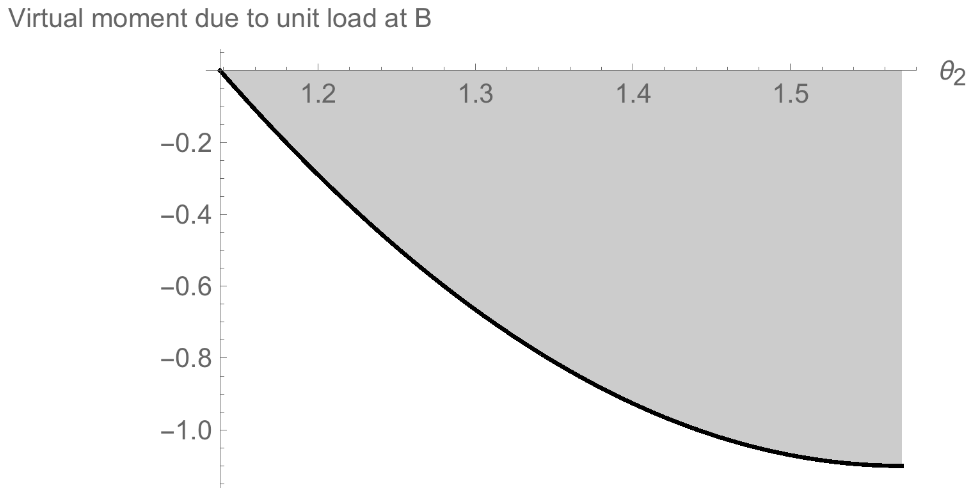
Virtual work for displacement at B

Again we place a unit load at the location where want to determine the displacement, yielding the following bending moment diagram (with alternative naming M_B used later):



To carry out the integration of $\left(\delta M\ \frac{M}{EI}\right)$ we express the bending moment in terms of y, which is earlier expressed in terms of θ_2 (tension at the top is considered positive):

```
virtualMomentB = -δF y;
```



The displacement at B is negative because the distributed load pulls upwards and hence "inward:"

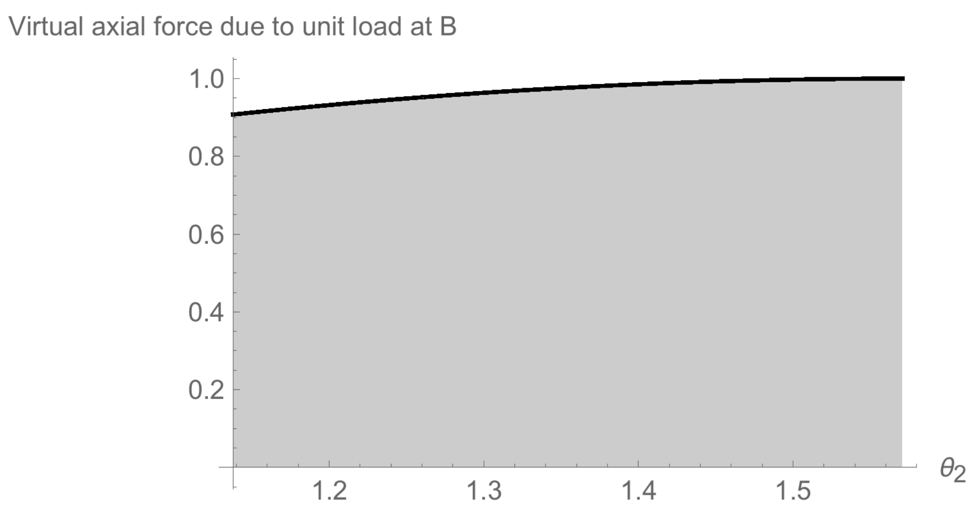
$$\triangle\text{BFlexure} = \frac{2}{\delta F}\int_{\theta 1}^{\frac{\pi}{2}}\left(\text{virtualMomentB}\ \frac{\text{bendingMoment}}{\text{E I}}\right)\text{R}\, d\theta 2;$$

```
UnitConvert[△BFlexure, "mm"]
```

which yields: −9.12801 mm

The virtual axial force is:

```
virtualAxialForceB = δF Sin[θ2];
```



The displacement due to axial deformation:

```
ΔBAxial = 2/δF ∫_θ1^(π/2) (virtualAxialForceB axialForce/(E A)) R dθ2;
UnitConvert[ΔBAxial, "mm"]
```

which yields: 0.00454041 mm

Flexibility method

We are now locking the support at B, making the beam statically indeterminate to the first degree. The new horizontal support reaction is considered as the redundant. It is determined by establishing the compatibility equation, with coefficients calculated by virtual work, combining internal forces determined above. In fact, the displacement at B due to external forces is already determined:

```
ΔB0 = ΔBFlexure + ΔBAxial;
UnitConvert[ΔB0, "mm"]
```

which yields: −9.12347 mm

The displacement at B due to a unit force at B is:

```
ΔBB = 2/δF^2 ∫_θ1^(π/2) (virtualMomentB^2/(E I)) R dθ2 + 2/δF^2 ∫_θ1^(π/2) (virtualAxialForceB^2/(E A)) R dθ2;
UnitConvert[ΔBB, "mm/kN"]
```

which yields: 0.423604 mm/kN

Solving for the redundant at B naturally reveals a reaction force toward the right because there load is acting upwards. This is the reaction force that makes the curved beam an arch when there is downward load:

```
XB = −ΔB0/ΔBB
```

which yields: 21.5377 kN

The final bending moment in the statically indeterminate arch is $M = M_0 + M_B X_B$:

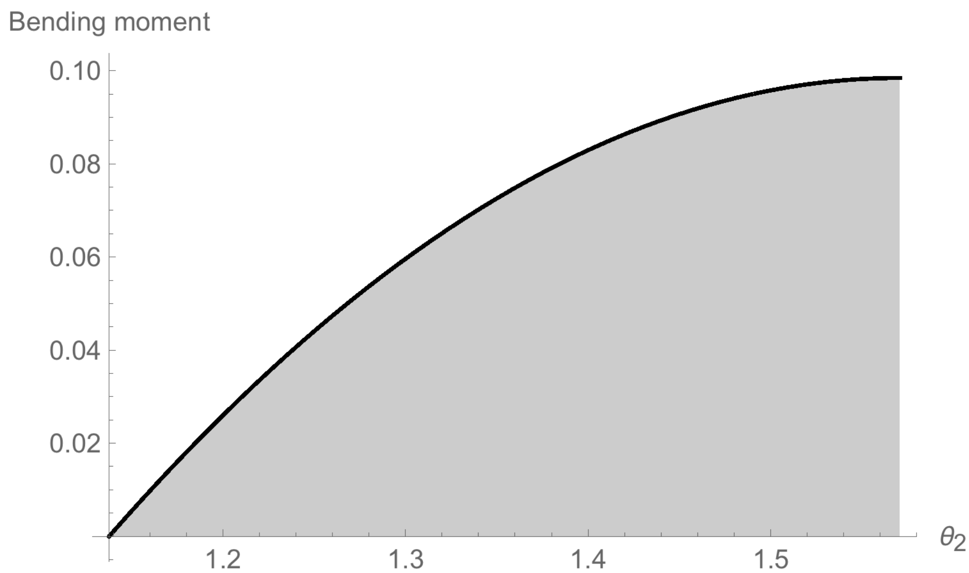
```
finalMoment = bendingMoment + virtualMomentB XB/δF;
```

The bending moment at midspan is:

```
finalMoment /. θ2 → π/2
```

which yields: 0.0984754 m kN

Here is a plot of the bending moment diagram on the left-hand side of midspan (tension at the top is considered positive):



The final axial force is $N = N_0 + N_B X_B$:

```
finalAxialForce = axialForce + virtualAxialForceB XB/δF;
```

The axial force at support A is:

```
finalAxialForce /. θ2 → θ1
```

which yields: 23.746 kN

The axial force at the midspan apex is:

```
finalAxialForce /. θ2 → π/2
```

which yields: 23.7377 kN

Notice that the axial force for an entire circle subjected to uniform "pressure" is:

```
q1 R
```

which yields: 23.8273 kN

Here is a plot of the axial force diagram on the left-hand side of midspan: