Theory of Elasticity, Article 18, Generic 2D Stress Distributions

One approach to solve 2D continuum problems analytically is to use "stress functions." Article 18 of the third edition of Theory of Elasticity by Timoshenko & Goodier, published in 1969 by McGraw-Hill, contains several versions of *polynomial* stress functions, Φ, all satisfying the governing equation

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\,\frac{\partial^4 \Phi}{\partial x^2\, y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

although some with certain conditions on the constant coefficients in Φ. In the following the stress functions are visualized to understand the stress pattern implied by each of them. In this document the notation and axis directions are somewhat different than in the book. Each section below displays the following items:

- Stress function, Φ
- Axials stress in x-direction, σ_{xx}
- Axial stress in y-direction, σ_{yy}
- Shear stress, τ_{xy}
- Sometimes a plot to visualize the stresses

Stress function x^2 (Constant σ_{yy})

$\Phi = C_{x2}\, x^2;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: 0

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: $2\, C_{x2}$

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: 0

Stress function y^2 (Constant σ_{xx})

$\Phi = C_{y2}\, y^2;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: $2\, C_{y2}$

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: 0

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: 0

Stress function xy (Constant τ_{xy})

$\Phi = C_{xy}\, x\, y;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: 0

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: 0

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: $-C_{xy}$

Stress function x^3 (Linearly varying σ_{yy} along x-axis)

$\Phi = C_{x3}\, x^3;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: 0

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: $6\, x\, C_{x3}$

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: 0

Stress function $x^2\, y$ (Linearly varying σ_{yy} and τ_{xy})

$\Phi = C_{x2y}\, x^2\, y;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: 0

$\sigma_{yy} = \partial_x \partial_x \Phi$

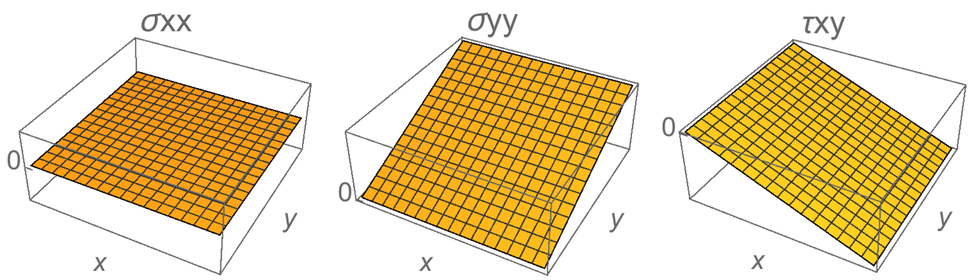
which yields: $2\, y\, C_{x2y}$

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: $-2\, x\, C_{x2y}$

```
plotValues = {Cx2y → 1};
σxxPlot = Plot3D[σxx /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σxx];
σyyPlot = Plot3D[σyy /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σyy];
σxyPlot = Plot3D[τxy /. plotValues, {x, 0, 100}, {y, 0, 100},
  AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → τxy];
GraphicsGrid[{{σxxPlot, σyyPlot, σxyPlot}}]
```

which yields:



Stress function xy^2 (Linearly varying σ_{xx} and τ_{xy})

$\Phi = C_{xy2}\, x\, y^2;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: $2\, x\, C_{xy2}$

$\sigma_{yy} = \partial_x \partial_x \Phi$

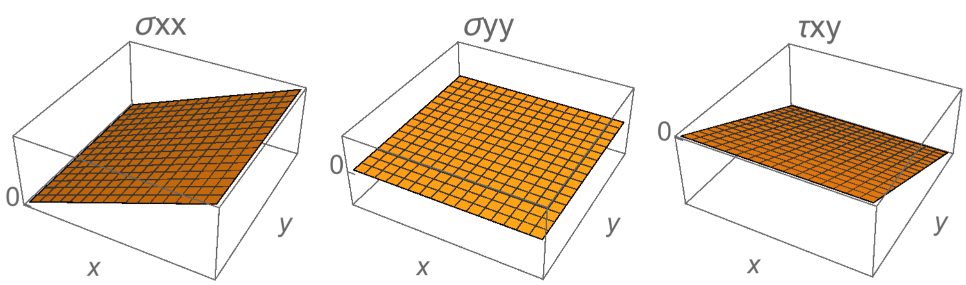
which yields: 0

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: $-2\, y\, C_{xy2}$

```
plotValues = {C_xy2 → 1};
σxxPlot = Plot3D[σ_xx /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σxx];
σyyPlot = Plot3D[σ_yy /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σyy];
σxyPlot = Plot3D[τ_xy /. plotValues, {x, 0, 100}, {y, 0, 100},
  AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → τxy];
GraphicsGrid[{{σxxPlot, σyyPlot, σxyPlot}}]
```

which yields:



Stress function y^3 (Linearly varying σ_{xx} along y-axis)

$\Phi = C_{y3}\, y^3;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: $6\, y\, C_{y3}$

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: 0

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: 0

Stress function with x^4 and $x^2\, y^2$

When using stress functions with fourth-order terms it is no longer guaranteed that the differentiation equation is satisfied. For example, by itself the term x^4 by itself does not work. This problem is mitigated by adding other fouth-order terms. For example:

$\Phi = C_{x4}\, x^4 + C_{x2y2}\, x^2\, y^2;$

This stress function satisfies the differential equation under this condition:

$\text{solution} = \text{Solve}[\partial_x \partial_x \partial_x \partial_x \Phi + 2\, \partial_x \partial_x \partial_y \partial_y \Phi + \partial_y \partial_y \partial_y \partial_y \Phi == 0]$

which yields: $\left\{\left\{C_{x4} \to -\frac{C_{x2y2}}{3}\right\}\right\}$

When that condition is satisfied the stresses are:

$\sigma_{xx} = \partial_y \partial_y \Phi\ /.\ \text{solution}[[1]]$

which yields: $2\, x^2\, C_{x2y2}$

$\sigma_{yy} = \partial_x \partial_x \Phi\ /.\ \text{solution}[[1]]$

which yields: $-4\, x^2\, C_{x2y2} + 2\, y^2\, C_{x2y2}$

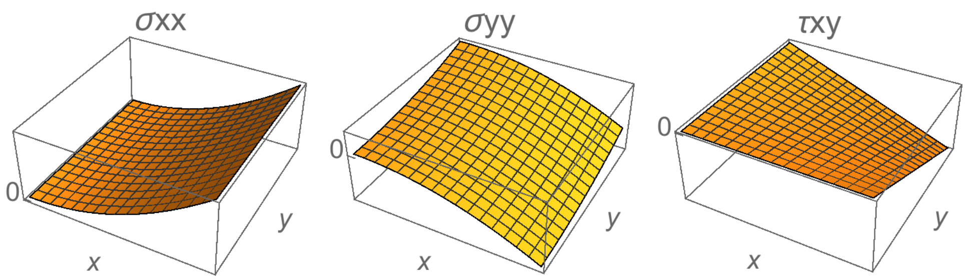
$\tau_{xy} = -\partial_x \partial_y \Phi\ /.\ \text{solution}[[1]]$

which yields: $-4\, x\, y\, C_{x2y2}$

Those expressions and the following plots shows that, e.g., σ_{xx} varies parabolically with x.

```
plotValues = {Cx2y2 → 1};
σxxPlot = Plot3D[σxx /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σxx];
σyyPlot = Plot3D[σyy /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σyy];
σxyPlot = Plot3D[τxy /. plotValues, {x, 0, 100}, {y, 0, 100},
  AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → τxy];
GraphicsGrid[{{σxxPlot, σyyPlot, σxyPlot}}]
```

which yields:



Stress function $x^3\ y$

$\Phi = C_{x3y}\ x^3\ y;$

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: 0

$\sigma_{yy} = \partial_x \partial_x \Phi$

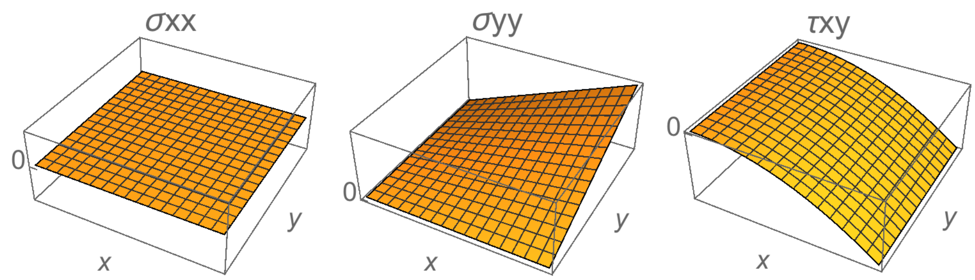
which yields: $6\ x\ y\ C_{x3y}$

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: $-3\ x^2\ C_{x3y}$

```
plotValues = {Cx3y → 1};
σxxPlot = Plot3D[σxx /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σxx];
σyyPlot = Plot3D[σyy /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σyy];
σxyPlot = Plot3D[τxy /. plotValues, {x, 0, 100}, {y, 0, 100},
  AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → τxy];
GraphicsGrid[{{σxxPlot, σyyPlot, σxyPlot}}]
```

which yields:



Stress function xy^3

$\Phi = C_{xy3}\, x\, y^3$;

$\sigma_{xx} = \partial_y \partial_y \Phi$

which yields: $6\, x\, y\, C_{xy3}$

$\sigma_{yy} = \partial_x \partial_x \Phi$

which yields: 0

$\tau_{xy} = -\partial_x \partial_y \Phi$

which yields: $-3\, y^2\, C_{xy3}$

```
plotValues = {Cxy3 → 1};
σxxPlot = Plot3D[σxx /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σxx];
σyyPlot = Plot3D[σyy /. plotValues, {x, 0, 100}, {y, 0, 100},
   AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → σyy];
σxyPlot = Plot3D[τxy /. plotValues, {x, 0, 100}, {y, 0, 100},
  AxesLabel → {x, y}, Ticks → {{}, {}, {0}}, PlotLabel → τxy];
GraphicsGrid[{{σxxPlot, σyyPlot, σxyPlot}}]
```

which yields: