# The St. Petersburg Paradox

Nicolaus Bernoulli (born 1687) first stated this problem in a letter to Pierre Raymond de Montmort in 1713 and his cousin Daniel Bernoulli presented a solution in 1738. The problem describes a situation where expected cost alone as decision criterion can give strange results. Suppose you are offered a game that involves repeatedly tossing a fair coin. Every time the coin is tossed there are two possible outcomes: heads and tails. Before starting the game you must pay a fixed amount to participate, and there is a pot of money (the stakes) that you win once the first tail appears. Importantly, the stakes start at $1 and double at every coin toss until the first tail appears and you win. According to the expected cost decision criterion, how much should you be willing to pay to participate in this game? The answer poses a paradox...

## Expected value of the game

The amount we will win is a discrete random variable. The possible values of the amount we will win is $1, $2, $4, $8, etc. This continues to infinity, although there is little chance that the game will go on for very long without a tail appearing. To compute the expectation, we need the probability associated with each possible win. With a fair coin, the probability of $1 is 0.5. The probability of $2, i.e., the probability of heads in botht the first and the second throw of the dice, is, assuming independence, (0.5)(0.5). In short, the expected value of the win is:

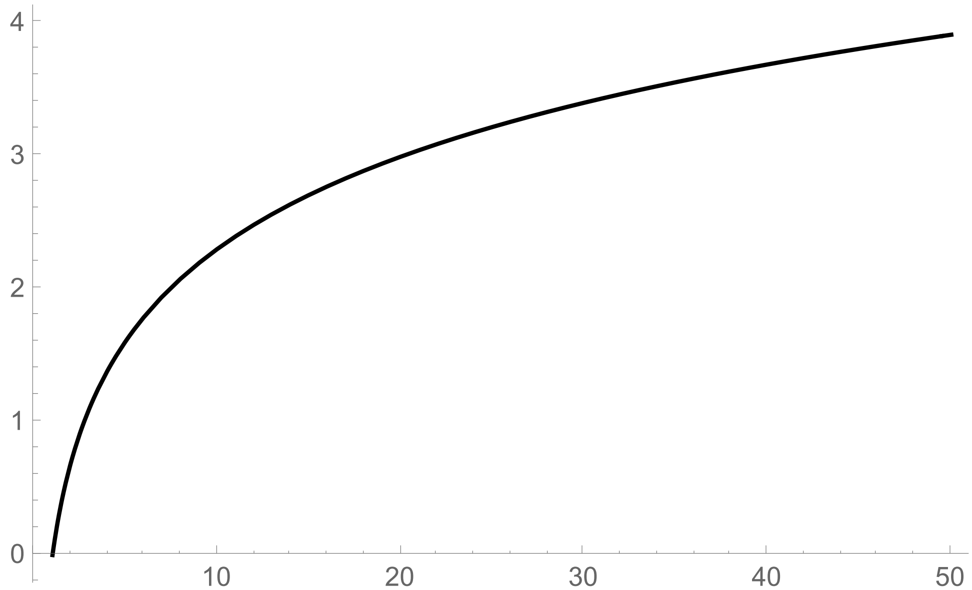
$$E(w) = \sum_{i=1}^{N} p_i w_i = \frac{1}{2}1 + \frac{1}{4}2 + \frac{1}{8}4 + \frac{1}{16}8 + \ldots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

It is seen that the expected value of the game is infinity. Thus, in accordance with the expected cost decision criterion we should be willing to pay an infinite amount of money to enter the game. That is a paradox, because our payment is wiped out once a heads appear, which will probably happen sooner rather than later.

## Solution to the paradox

The classical solution by Bernoulli is to introduce a utility function to express the "diminishing marginal utility" of money. According to Bernoulli: "There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man although both gain the same amount." Bernoulli suggested the logarithmic utility function u(w)=ln(w) to relate money and utility:

```
Plot[Log[w], {w, 1, 50}, PlotStyle → Black]
```



Then, the expected utility is:

$$E(u) = \sum_{i=1}^{N} p_i u_i = \frac{1}{2}\text{Log}[1] + \frac{1}{4}\text{Log}[2] + \frac{1}{8}\text{Log}[4] + \frac{1}{16}\text{Log}[8] + \ldots$$

Evaluate the sum:

$$\text{expectedUtility} = \sum_{i=1}^{\infty} \frac{\text{Log}\left[2^{i-1}\right]}{2^{i}};$$

Display the answer analytically:

```
expectedUtility
```

which yields: $\sum_{i=1}^{\infty} 2^{-i}\,\text{Log}\left[2^{-1+i}\right]$

Display the answer as a real number:

```
expectedUtility // N
```

which yields: 0.693147

Given the relationship utility=ln(dollar), that utility value corresponds to the following dollar value:

```
expectedValue = Exp[expectedUtility]
```

which yields: $e^{\sum_{i=1}^{\infty} 2^{-i} \mathrm{Log}[2^{-1+i}]}$

In words, we should be willing to pay $2 to participate in the game with this utility function.

## Player wealth

It is also possible to include the wealth of the player, denoted *B* for bankroll below. Different players will have different bankrolls, which will influence their decision. Now we formulate the expected CHANGE in utility, i.e., utility AFTER the game minus utility BEFORE the game, including the cost of participating in the game, C.

```
B = 1000;
```

$$\sum_{i=1}^{\infty} \frac{\mathrm{Log}[B + 2^{i-1} - 5.94] - \mathrm{Log}[B]}{2^{i}} \; // \; N$$

which yields: 0.0000280369

For example, with log utility a millionaire should be willing to pay up to $10.94, a person with $1000 should pay up to $5.94, a person with $2 should pay up to $2, and a person with $0.60 should borrow $0.87 and pay up to $1.47.