Stress and Strain

According to modern structural engineering structures break when the stress, or a stress resultant, exceed a limiting value. In other words, stress is today the key design criterion. In addition the strain, or resulting deformation, is often considered in order to avoid uncomfortably large deflections.

It is interesting to note that the beam theory we today name after Leonard Euler (1707-1783) and Daniel Bernoulli (1700-1782), and to some extent Daniel's uncle Jacob Bernoulli (1654-1705), came about before the concepts of stress and strain existed. While Euler and Bernoulli established the beam theory with the correct neutral axis it was Augustin Cauchy (1789-1857) who first introduced the concepts of stress and strain around 1822. Although Antoine Parent (1666-1716) and later Charles-Augustin de Coulomb (1736-1806) used similar concepts in beam theory, earlier work essentially considered the molecular forces between individual particles rather than stress distributed in a solid continuum. Cauchy took the continuum idea from hydrodynamics (Timoshenko 1953) and the concepts stress and strain in solids are now omnipresent in solid mechanics and structural engineering.

In Cauchy's idealization, each material particle is subjected to stresses that result in strains, or conversely, strains that result in stresses. In practical applications, for example beam bending, we are usually interested in finding stresses, strains, and deformations due to some applied load. Alternatively, we may be after stresses, strains, and deformations due to an imposed displacement. Regardless, this is formally a boundary value problem (BVP) that requires boundary conditions in addition to three sets of equations:

- Material law equations that relate stresses with strains
- Equilibrium equations that relate stresses with external forces
- Kinematic compatibility equations that relate strains with global deformations

Many of the documents on this website formulate such equations for different types of problems, such as beams, trusses, and plates. This document introduces the concepts and notation. To that end, there are two types of stress: axial stress and shear stress. Often σ=axial stress and τ=shear stress, but sometimes σ is used for both. Similarly, ε=axial strain and γ=shear strain, but sometimes ε is used for both. However, while σ and τ are interchangeable in the notation of shear stress, $\gamma \neq \varepsilon$ in the notation of shear strain where $\gamma = 2\varepsilon$. This will be understood when considering equations for kinematic compatibility.

Figure 1 shows the stress components for an infinitesimally small cube of a continuum material. All stresses act in a coordinate direction hence these are called coordinate stresses. The first index indicates the direction of the normal vector to the plane where the stress acts and the second index indicates the direction of the stress. Axial stresses are positive in tension and shear stresses are positive when they act in the positive axis direction on a surface that has a positive axis direction as the surface normal. In Figure 1 σ=axial stress and τ=shear stress and when σ is used for both components with equal indices are understood to be axial stresses and components with different indices are shear stresses. The coordinate stresses are collected in the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{1}$$

written σ_{ij} in index notation. Similarly the strain tensor is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0.5\cdot\gamma_{xy} & 0.5\cdot\gamma_{xz} \\ 0.5\cdot\gamma_{yx} & \varepsilon_{yy} & 0.5\cdot\gamma_{yz} \\ 0.5\cdot\gamma_{zx} & 0.5\cdot\gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{2}$$

written ε_{ij} in index notation.

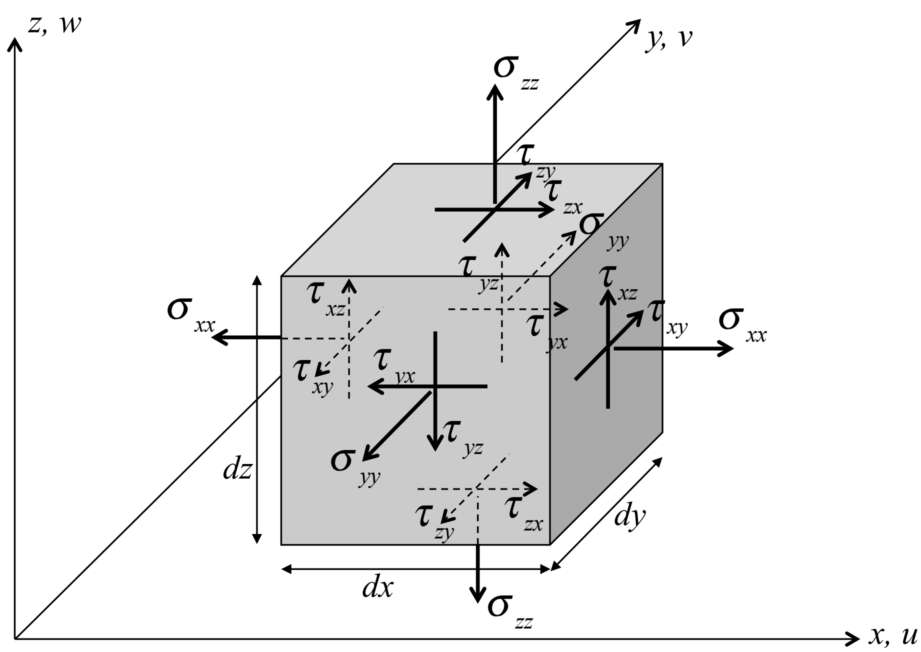

Figure 1: Stress components.

Sometimes the stress and strain tensors are written in Voight notation:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{3}$$

References

Timoshenko, S. P. (1953). *History of strength of materials*. McGraw-Hill.