Stress Transformations

Other documents on this website address coordinate stresses, i.e., stress components along the coordinate axes. Equations for equilibrium, material law, and kinematic compatibility are established for coordinate stresses and strains. However, it is useful to transform the stresses, and sometimes the strains, into other coordinate system. This means that we seek stresses on planes that are not aligned with the coordinate axes. One useful outcome is the determination of "principal stresses" as explained later.

2D Transformations

Consider the plane stress state, in which only σ_{xx}, σ_{yy}, and $\sigma_{xy}=\sigma_{yx}\equiv\tau_{xy}$ act. Suppose that these coordinate stresses are known. The objective in this section is to determine the stress state in rotated configurations, *e.g.*, to determine the principal stresses. Let θ denote the angle (positive counter-clockwise) between the original coordinate system and the rotated one. The rotated plane is shown in Figure 1, where the stresses on that plane are called σ and τ.

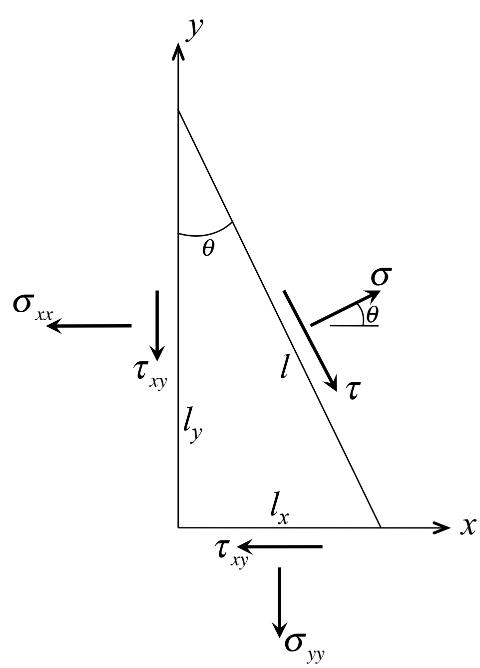

Figure 1: Stresses on an inclined plane.

By noting that $\cos(\theta)=l_y/l$ and $\sin(\theta)=l_x/l$, equilibrium in the direction of σ yields

$$\sigma = \sigma_{xx} \cdot \cos^2(\theta) + \sigma_{yy} \cdot \sin^2(\theta) + 2 \cdot \tau_{xy} \cdot \cos(\theta) \cdot \sin(\theta) \tag{1}$$

The trigonometric identities $\cos^2(\theta)=(1+\cos(2\theta))/2$, $\sin^2(\theta)=(1+\sin(2\theta))/2$, and $\sin(2\theta)=2\sin(\theta)\cos(\theta)$ lead to the modified expression

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \cos(2\theta) + \tau_{xy} \cdot \sin(2\theta) \tag{2}$$

Similarly, equilibrium in the direction of τ yields:

$$\tau = \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \sin(2\theta) - \tau_{xy} \cdot \cos(2\theta) \tag{3}$$

Eqs. (2) and (3) establish the basis for transformation (rotation) of stresses in two-dimensional stress states. Extreme values of σ and τ and the corresponding angle θ are determined by setting the derivative of Eqs. (2) and (3) with respect to θ equal to zero. However, the graphical approach known as Mohr's circle is an appealing alternative to analytical derivations.

Mohr's Circle

Eqs. (2) and (3) represent a circle in the σ–τ plane. To derive the expression for the circle, move the first term in the right-hand side of Eq. (2) to the left-hand side. Then square Eqs. (2) and (3) and add them. Upon using the trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$ and cancelling terms, one obtains

$$\left(\sigma - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 \tag{4}$$

This equation forms a circle in the σ–τ plane, shifted along the σ-axis, as shown in Figure 2. All points on Mohr's circle represents stress states at planes of different angle θ. In fact, drawing the circle immediately reveals the maximum and minimum axial stresses at the locations with zero shear stress. Conversely, it is observed in Figure 2 that the stress states with maximum shear stress are usually *not* associated with zero axial stresses.

Although there are several methods for working with Mohr's circle, two questions appear prominently in its practical use: What is the orientation, θ, of the plane for any of the stress states on the circle? What is the direction of the shear stress at any point on the circle? These questions are answered by the following procedure:

1. Draw Mohr's circle with the radius and centre-shift as shown in Figure 2.
2. On the circle, identify the location of the stress state (σ_{xx}, τ_{xy}) that acts on the right-hand side of the particle, i.e., on the plane that has the *x*-axis as the surface normal. This point is called the "origin of planes." By definition, this stress state is associated with θ=0. Be careful with the sign of the shear stress; clockwise shear stress is positive, which contradicts the positive direction of the coordinate stress on this surface.
3. Draw a horizontal line from the origin of planes. The point where the line intersects with the circle is called the "pole point." When $\sigma_{xx}=\sigma_{yy}$ the origin of planes coincides with the pole point.
4. Draw a straight line from the pole point to any point of Mohr's circle and study that stress state:

a. The abscissa axis provides the axial stress (tension is positive).
b. The ordinate axis provides the shear stress (clockwise is positive).
c. The angle between the horizontal axis and that straight line equals θ (see the identification of θ in Figure 2).

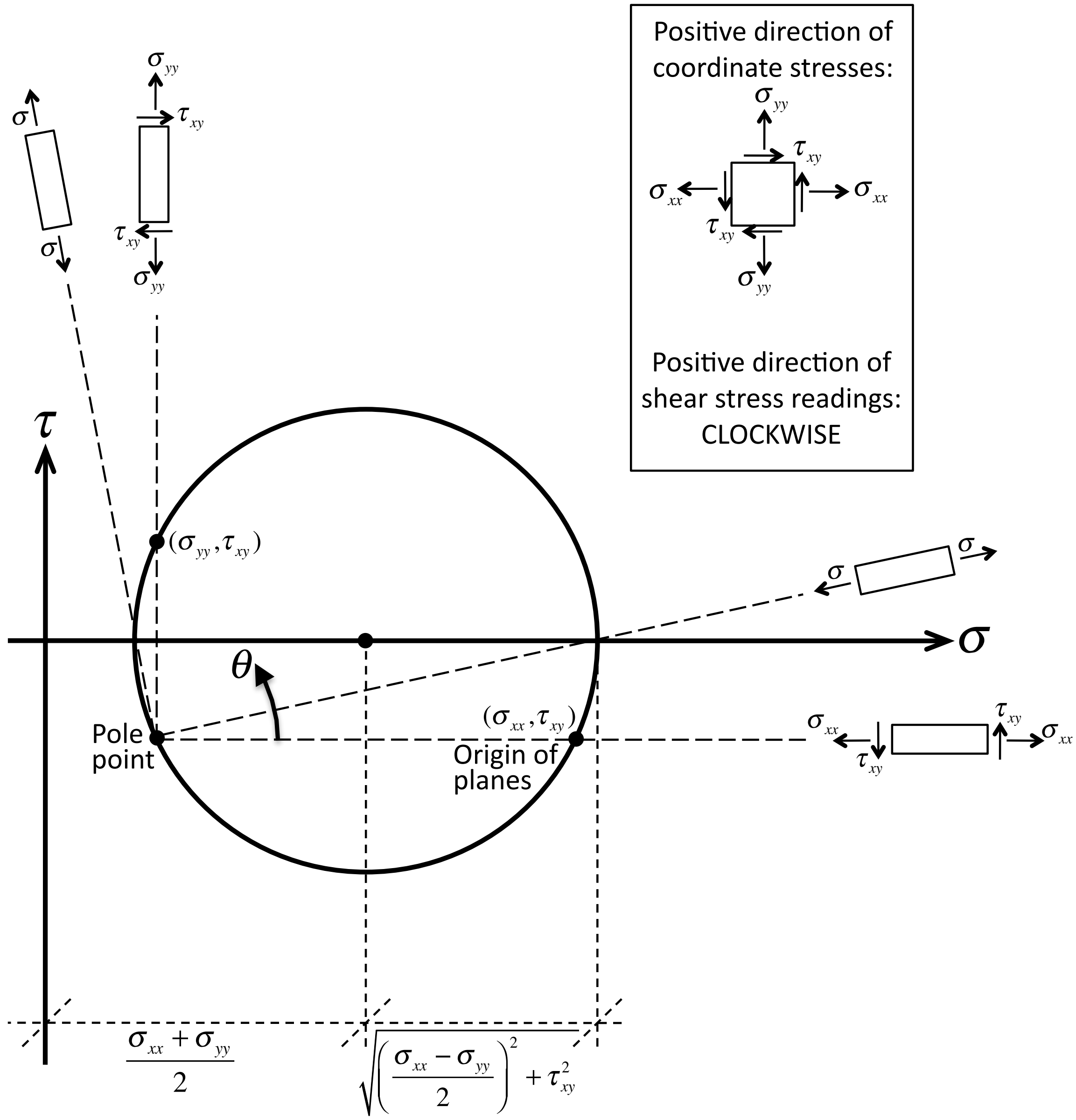

Figure 2: Mohr's circle.

Notice in particular that the stress state (σ_{yy}, τ_{xy}) is confirmed by drawing a vertical line from the pole point. This is the stress state that acts on the plane that has the *y*-axis as the surface normal. By definition, this stress state is associated with θ=90^o but on the circle it will always be 180^o away from the origin of planes. Also, notice that the direction of the principal axes are identified by drawing a straight line from the pole point to the locations on the circle with zero shear stress, as shown in Figure 2.

For practical purposes it is useful to synthesize the earlier results for the determination of extreme stress values. Inspection of Mohr's circle for the 2D case shows that the maximum axial stress is

$$\sigma_1 = \max\left\{0 \quad , \quad \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right\} \tag{5}$$

and that the minimum axial stress is

$$\sigma_3 = \min\left\{0 \quad , \quad \frac{\sigma_{xx}+\sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}\right\} \tag{6}$$

In this document the symbols σ_1 and σ_3 are reserved for the maximum and minimum stress, respectively. This explains the inclusion of zero as a possibility in Eqs. (5) and (6). This notation requires particular attention in 2D stress situations, where the out-of-plane stress is zero and, thus, often equals σ_3. For the 2D case, Mohr's circle also shows that the maximum shear stress equals the radius of the circle:

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{7}$$

3D Transformations

Consider the stress traction $\mathbf{t}=\{t_x\ t_y\ t_z\}^T$ that acts on an infinitesimal surface area with surface normal $\mathbf{n}=\{n_x\ n_y\ n_z\}^T$ shown in Figure 3, where t_i is the force in the *i*-direction. Let dA denote the area of the inclined surface on which the traction acts and let dA_x denote the area of the side that has the negative *x*-axis as normal vector, and so forth. Equilibrium in the *x*-direction yields:

$$t_x \cdot dA = \sigma_{xx} \cdot dA_x + \sigma_{yx} \cdot dA_y + \sigma_{zx} \cdot dA_z \tag{8}$$

To refine the expression, consider the relationship between the areas dA and dA_i. Figure 3 shows that dA=0.5hl and dA_z=0.5$h_z l$. Consequently,

$$\frac{dA_z}{dA} = \frac{h_z}{h} = \cos(\theta_z) = \cos(z,\mathbf{n}) = n_z \tag{9}$$

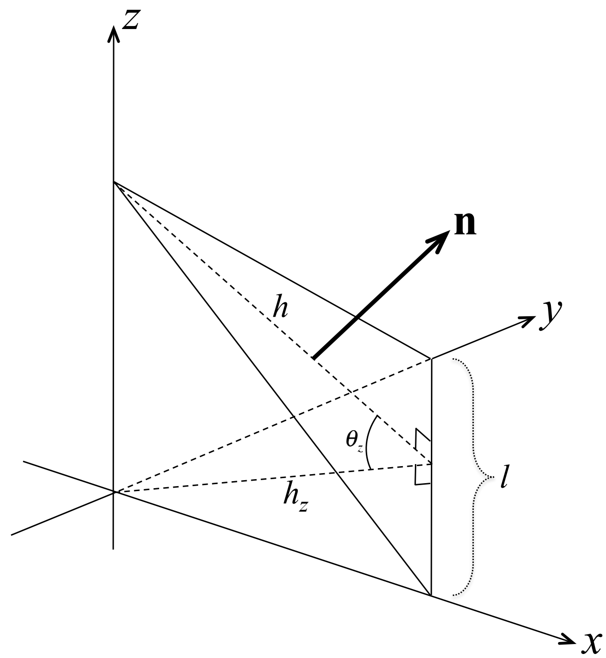

Figure 3: Surface on which the stress traction acts.

Dividing Eq. (8) by *dA* yields

$$t_x = \sigma_{xx} \cdot n_x + \sigma_{yx} \cdot n_y + \sigma_{zx} \cdot n_z \tag{10}$$

Repeating this exercise for all three axis-directions, and noting that $\sigma_{ij}=\sigma_{ji}$ because of equilibrium considered later in this document, yields the equilibrium equations due to Cauchy that relate a surface traction to the coordinate stresses:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = t_i = \sigma_{ij} n_j \tag{11}$$

It is noted that, because **n** is a unit vector, the axial stress on a plane with normal vector **n** is the dot product between **n** and the stress traction:

$$\sigma_{\mathbf{n}} = \mathbf{t}^T \mathbf{n} \tag{12}$$

Subsequently, the Pythagorean theorem determines the largest shear stress on the plane:

$$\tau_{\mathbf{n}} = \sqrt{\|\mathbf{t}\|^2 + \sigma_{\mathbf{n}}^2} \tag{13}$$

Principal Stresses

For every material particle it is possible to find three orthogonal planes with zero shear stresses. The axial stresses on these planes are called principal stresses. They represent the extreme values of the stresses, where σ_1 is the largest and σ_3 is the smallest principal stress (in absolute value). One approach to determine the principal stresses is to employ Eq. (11). For a plane with principal stresses the traction vector is parallel with the normal vector of that plane; *i.e.*, there are no shear stresses on that plane and the traction is the scaled normal vector:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \lambda \cdot \mathbf{n} \tag{14}$$

This is an eigenvalue problem in the unknown scalar λ, *i.e.*, $(\boldsymbol{\sigma} - \lambda \cdot \mathbf{I})\mathbf{n} = 0$. Solutions are obtained by setting the determinant of the coefficient matrix equal to zero:

$$\lambda^3 - I_1 \cdot \lambda^2 + I_2 \cdot \lambda - I_3 = 0 \tag{15}$$

where the stress variants are defined as

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{16}$$

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} \tag{17}$$

$$I_3 = |\boldsymbol{\sigma}| \tag{18}$$

where vertical bars indicate the determinant operation. Upon solving for the eigenvalues, λ, i.e., σ_1, σ_2, and σ_3, the eigenvectors yield the principal directions. The quantities I_1, I_2, and I_3 are called stress invariants because they retain the same value regardless of the orientation of the coordinate system. These stress invariants are somewhat different from the stress invariants J_1, J_2, and J_3 that are mentioned in the section below on stress-based failure criteria and extensively used in the theory of plasticity, which is described in another document on material nonlinearity. Note that the maximum shear stress in 3D is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{19}$$