Response Sensitivity Analysis

Suppose the displacement vector, $\mathbf{u}$, containing displacements and rotations along all degrees of freedom, has been obtained by a finite element analysis. That response vector depends on a number of input parameters for loads, materials, and geometry. All those parameters are collected in the vector $\boldsymbol{\theta}$. The objective in this document is to calculate the derivatives, $\partial\mathbf{u}/\partial\boldsymbol{\theta}$, called the gradient vector or simply response sensitivities. The derivative of other responses, such as stress and strain, can be obtained once $\partial\mathbf{u}/\partial\boldsymbol{\theta}$ is calculated. Response sensitivities are useful because they reveal how sensitive the response is to variations in input values. The gradient vector is also needed in gradient-based reliability and optimization analyses. Finite difference schemes represent one approach for determining $\partial\mathbf{u}/\partial\boldsymbol{\theta}$, implying that the components of $\boldsymbol{\theta}$ are varied and the finite element analysis is re-run, one parameter at a time. This approach is problematic because the results are approximate and it can be sometimes be challenging to select a reasonable parameter perturbation value. Conversely, the direct differentiation method is employed in this document. Here the results are exact at the one-time cost of implementing equations for response derivatives inside the finite element code.

Linear Static Analysis

The classical stiffness method, i.e., linear elastic finite element analysis, provides an instructive introduction to the direct differentiation method. Consider the governing equilibrium equations in matrix form, in which the dependence on load-, material-, and geometry-parameters is made clear:

$$\mathbf{K}(\boldsymbol{\theta})\mathbf{u}(\boldsymbol{\theta}) = \mathbf{F}(\boldsymbol{\theta}) \tag{1}$$

Differentiating Eq. (1) through with respect to $\boldsymbol{\theta}$, using the product rule of differentiation, yields

$$\frac{\partial \mathbf{K}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\mathbf{u}(\boldsymbol{\theta}) + \mathbf{K}(\boldsymbol{\theta})\frac{\partial \mathbf{u}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \tag{2}$$

Solving for $\partial\mathbf{u}/\partial\boldsymbol{\theta}$ yields the desired result:

$$\frac{\partial \mathbf{u}}{\partial \boldsymbol{\theta}} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}}\mathbf{u}\right) \tag{3}$$

Remembering that the ordinary response $\mathbf{u}$ is calculated by $\mathbf{u}=\mathbf{K}^{-1}\mathbf{F}$ it is observed that the response sensitivities are obtained from a similar equation, only with a different right-hand side. It turns out that this observation in response sensitivity analysis holds for even inelastic and dynamic analysis: once the ordinary response calculations have converged then the derivatives are obtained by solving a linear system of equations.

Inelastic Static Analysis

With inelastic material behaviour the governing equilibrium equations are written

$$\mathbf{P}_{\text{int}}\left(\mathbf{u}(\boldsymbol{\theta}),\boldsymbol{\theta}\right)=\mathbf{P}_{\text{ext}}(\boldsymbol{\theta}) \tag{4}$$

where $\mathbf{P}_{\text{int}}$=internal forces, which are $\mathbf{Ku}$ in linear elastic analysis, $\mathbf{P}_{\text{ext}}$=external forces, written $\mathbf{F}$ in the classical stiffness method. It is shown in Eq. (4) that $\mathbf{P}_{\text{int}}$ depends on $\mathbf{u}$ and hence implicitly on $\boldsymbol{\theta}$, but also explicitly on $\boldsymbol{\theta}$. Hence, the chain rule of differentiation is used in the differentiation of Eq. (4), which yields

$$\frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \boldsymbol{\theta}}+\left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \boldsymbol{\theta}}\right|_{\mathbf{u}\text{ fixed}}=\frac{\partial \mathbf{P}_{\text{ext}}}{\partial \boldsymbol{\theta}} \tag{5}$$

The first term in the left-hand side of Eq. (5) accounts for the implicit variation of $\mathbf{P}_{\text{int}}$ on $\boldsymbol{\theta}$, while the second term accounts for the potential explicit variation of $\mathbf{P}_{\text{int}}$ on $\boldsymbol{\theta}$. The sought response sensitivities are now obtained by solving Eq. (5):

$$\frac{\partial \mathbf{u}}{\partial \boldsymbol{\theta}}=\mathbf{K}_{\text{tangent}}^{-1}\left(\frac{\partial \mathbf{P}_{\text{ext}}}{\partial \boldsymbol{\theta}}-\left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \boldsymbol{\theta}}\right|_{\mathbf{u}\text{ fixed}}\right) \tag{6}$$

where the tangent stiffness matrix is

$$\mathbf{K}_{\text{tangent}}^{-1}=\frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{u}} \tag{7}$$

Eq. (6) shows that response sensitivities in static inelastic analysis are obtained from a linear system of equations that is solved once the response $\mathbf{u}$ has been obtained from the iterative procedure to solve Eq. (4), involving the repeated solution of an equation similar to Eq. (6). In fact, an advantage of using the Newton-Raphson method to solve for the response u is here observed: the tangent stiffness used in the last step of the Newton-Raphson method is required to solve for the response sensitivities. Another observation about Eq. (6) is that $\partial \mathbf{P}_{\text{int}}/\partial \boldsymbol{\theta}$ for fixed $\mathbf{u}$ requires the differentiation of all equations involved in calculating $\mathbf{P}_{\text{int}}$. It is often this differentiation of equations within a material model or within an element formulation that is the challenging part of implementing the direct differentiation method.

Linear Dynamic Analysis

When including inertial and viscous forces the governing equilibrium equations read

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\mathbf{u}}(\boldsymbol{\theta})+\mathbf{C}(\boldsymbol{\theta})\dot{\mathbf{u}}(\boldsymbol{\theta})+\mathbf{K}(\boldsymbol{\theta})\mathbf{u}(\boldsymbol{\theta})=\mathbf{F}(\boldsymbol{\theta}) \tag{8}$$

where $\mathbf{M}$=mass matrix, $\mathbf{C}$=damping matrix, and each dot over $\mathbf{u}$ represents differentiation with respect to time. In other words, in Eq. (8) the mass matrix is multiplied by the accelerations, while the velocities multiply the damping matrix. Before differentiating to obtain response sensitivities, $\partial \mathbf{u}/\partial \boldsymbol{\theta}$, the time-stepping scheme used to solve for the ordinary response, $\mathbf{u}$, is applied. The following general time-stepping scheme, which includes the popular Newmark and Wilson schemes, is convenient because it covers many options:

$$\ddot{\mathbf{u}}_{n+1}=a_1\cdot\mathbf{u}_{n+1}+a_2\cdot\mathbf{u}_n+a_3\cdot\dot{\mathbf{u}}_n+a_4\cdot\ddot{\mathbf{u}}_n \tag{9}$$

$$\dot{\mathbf{u}}_{n+1} = a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n \tag{10}$$

where n=time-step number and a_i=constant coefficients that determine the type of time-stepping scheme. Substitution of Eqs. (9) and (10) into Eq. (8) yields the following equilibrium equations at step n+1:

$$\begin{aligned} &\mathbf{M}\left(a_1 \cdot \mathbf{u}_{n+1} + a_2 \cdot \mathbf{u}_n + a_3 \cdot \dot{\mathbf{u}}_n + a_4 \cdot \ddot{\mathbf{u}}_n\right) \\ &+\mathbf{C}\left(a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n\right) + \mathbf{K}\mathbf{u}_{n+1} = \mathbf{F}_{n+1} \end{aligned} \tag{11}$$

It is noted that Eq. (11) is a linear system of equations for the unknown un+1 with coefficient matrix a_1**M**+a_5**C**+**K**. Differentiating Eq. (11) through with respect to $\boldsymbol{\theta}$ yields

$$\begin{aligned} &\frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\left(a_1 \cdot \mathbf{u}_{n+1} + a_2 \cdot \mathbf{u}_n + a_3 \cdot \dot{\mathbf{u}}_n + a_4 \cdot \ddot{\mathbf{u}}_n\right) \\ &+\mathbf{M}\left(a_1 \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} + a_2 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_3 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_4 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) \\ &+\frac{\partial \mathbf{C}}{\partial \boldsymbol{\theta}}\left(a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n\right) \\ &+\mathbf{C}\left(a_5 \cdot \frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} + a_6 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_7 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_8 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) + \frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}}\mathbf{u}_{n+1} + \mathbf{K}\frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{F}_{n+1}}{\partial \boldsymbol{\theta}} \end{aligned} \tag{12}$$

Rearranging yields a linear system of equations for the sought response sensitivities $\partial\mathbf{u}_{n+1}/\partial\boldsymbol{\theta}$ with the same coefficient matrix, a_1**M**+a_5**C**+**K**, as observed earlier in Eq. (11) for the ordinary response $\mathbf{u}_{n+1}$:

$$\begin{aligned} \left(a_1 \cdot \mathbf{M} + a_5 \cdot \mathbf{C} + \mathbf{K}\right)\frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} = &\frac{\partial \mathbf{F}_{n+1}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}}\mathbf{u}_{n+1} \\ &-\frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\left(a_1 \cdot \mathbf{u}_{n+1} + a_2 \cdot \mathbf{u}_n + a_3 \cdot \dot{\mathbf{u}}_n + a_4 \cdot \ddot{\mathbf{u}}_n\right) \\ &-\frac{\partial \mathbf{C}}{\partial \boldsymbol{\theta}}\left(a_5 \cdot \mathbf{u}_{n+1} + a_6 \cdot \mathbf{u}_n + a_7 \cdot \dot{\mathbf{u}}_n + a_8 \cdot \ddot{\mathbf{u}}_n\right) \\ &-\mathbf{M}\left(a_2 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_3 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_4 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) \\ &-\mathbf{C}\left(a_6 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_7 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_8 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) \end{aligned} \tag{13}$$

While Eq. (13) looks intimidating it is actually unproblematic and conceptually identical to Eq. (3), except with a larger right-hand side. The look of Eq. (13) can be somewhat improved by recognizing Eqs. (9) and (10) within it:

$$\left(a_1 \cdot \mathbf{M} + a_5 \cdot \mathbf{C} + \mathbf{K}\right)\frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{F}_{n+1}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}}\mathbf{u}_{n+1} - \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\ddot{\mathbf{u}}_{n+1} - \frac{\partial \mathbf{C}}{\partial \boldsymbol{\theta}}\dot{\mathbf{u}}_{n+1} - \mathbf{M}\left(a_2 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_3 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_4 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) - \mathbf{C}\left(a_6 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_7 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_8 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) \tag{14}$$

It is observed that Eq. (14) does NOT need to be solved at each time step in order to obtain response sensitivities at the last time step. It is sufficient to solve Eq. (14) at that last step. That fact changes in inelastic dynamic analysis, addressed shortly. Another observation is made in regards to $\partial\mathbf{C}/\partial\boldsymbol{\theta}$ in Eq. (14). This derivative is unproblematic when proportional Rayleigh damping is used, i.e., when the damping matrix consists of a portion of the stiffness matrix plus a portion of the mass matrix. However, if **C** depends on the displacements or velocities then a more careful treatment is needed and the linearity of Eq. (14) is no longer guaranteed.

Inelastic Dynamic Analysis

This most general case is governed by the equilibrium equations

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\mathbf{u}}(\boldsymbol{\theta}) + \mathbf{C}(\boldsymbol{\theta})\dot{\mathbf{u}}(\boldsymbol{\theta}) + \mathbf{P}_{\text{int}}\left(\mathbf{u}(\boldsymbol{\theta}), \dot{\mathbf{u}}(\boldsymbol{\theta}), \boldsymbol{\theta}\right) = \mathbf{P}_{\text{ext}}(\boldsymbol{\theta}) \tag{15}$$

where the internal forces' dependence on both displacements and velocities are included, implying the inclusion of viscous forces. Determination of response sensitivities in this case is a combination of the inelastic case in Eq. (5) and the dynamic case in Eq. (14), yielding

$$\left(a_1 \cdot \mathbf{M} + a_5 \cdot \mathbf{C} + \frac{\partial \mathbf{P}_{\text{int}}}{\partial \mathbf{u}}\right)\frac{\partial \mathbf{u}_{n+1}}{\partial \boldsymbol{\theta}} = \frac{\partial \mathbf{P}_{\text{ext},\, n+1}}{\partial \boldsymbol{\theta}} - \left.\frac{\partial \mathbf{P}_{\text{int}}}{\partial \boldsymbol{\theta}}\right|_{\mathbf{u}\text{ fixed}} - \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\ddot{\mathbf{u}}_{n+1} - \frac{\partial \mathbf{C}}{\partial \boldsymbol{\theta}}\dot{\mathbf{u}}_{n+1} - \mathbf{M}\left(a_2 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_3 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_4 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) - \mathbf{C}\left(a_6 \cdot \frac{\partial \mathbf{u}_n}{\partial \boldsymbol{\theta}} + a_7 \cdot \frac{\partial \dot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}} + a_8 \cdot \frac{\partial \ddot{\mathbf{u}}_n}{\partial \boldsymbol{\theta}}\right) \tag{16}$$

Again the system of equations for the response sensitivity $\partial\mathbf{u}_{n+1}/\partial\boldsymbol{\theta}$ looks a bit intimidating, but all quantities in the right-hand side are known and the coefficient matrix $a_1\mathbf{M}+a_5\mathbf{C}+\mathbf{K}$ is the same as the tangent stiffness used in Newton-Raphson iterations for the response itself. The same observation as was made earlier for damping is made here about the external forces, namely that they are "conservative" and assumed not to vary with the displacement response, which could make Eq. (16) nonlinear.

Derivative of Other Responses

When the derivative of stresses, strains, bending moments, etc. are needed then the equations that lead from the displacement response to those responses are differentiated. For example, to find the derivative of a bending moment with respect to an input parameter, the slope-deflection equation is differentiated. Similarly, to find the derivative of a stress in a plane finite element with respect to an input parameter, the displacement-to-stress equations involving the **D**-matrix, **B**-matrix, etc. are differentiated. This is not conceptually problematic but perhaps time-consuming the same way that the differentiation of inelastic material and element routines is. However, the advantage of the direct differentiation method is that exact response sensitivities are obtained in every finite element analysis when that work is done once and for all.