Random vibrations with Gaussian process

Units: kg, N, m

Consider a simply supported beam made of steel with Young's modulus E, density ρ, damping ξ. It has length L and the cross-section is solid recgangular with width b and height h. The beam is subject to a point load, F, at midspan modelled as a zero-mean Gaussian stochastic process with a one-sided power spectral density with constant height S0 from frequency $\omega 1$ to $\omega 2$. Use the following values:

```
b = 0.3 ;
h = 0.05;
L = 6;
E = 200 × 10^9;
ρ = 7850;
ξ = 0.03;
g = 9.81;
ω1 = 6;
ω2 = 8;
S0 = 50 000;
```

Is the load process narrowband or broadband?
What is the natural frequency of vibration and corresponding period for this beam?
What is the displacement at the midpoint of the beam due to one standard deviation of the load applied statically?
What is the response spectrum for the displacement at midspan, assuming a constant transfer function with amplitude sampled at the centre of the load spectrum?
Is the displacement response process narrowband or broadband?
What is the standard deviation of the displacement response?
What is the rate of crossing the displacement theshold r, and what is then the average time between such up-crossings?
What is the mean and standard deviation of the response peaks? Also plot the PDF.
What is the mean and standard deviation of the extremes within one hour? Also plot the CDF.
What is the probability that the extremes in that period will exceed 3 times r?
What is the threshold that has only 1% chance of being exceeded that period?
Consider the possibility of fatigue due to axial stress at midspan and suppose the SN-curve is characterized by M and K. What is the expected fatigue life, in days?

```
SNcurveK = 10^28;
SNcurveM = 3.0;
r = 0.004;
```

Spectral moments

$$\lambda 0 = \int_{\omega 1}^{\omega 2} \text{S0}\, d\omega$$

which yields: 100 000

$$\lambda 2 = \int_{\omega 1}^{\omega 2} \omega^2\, \text{S0}\, d\omega \; // \; \text{N}$$

which yields: 4.93333×10^6

$$\lambda 4 = \int_{\omega 1}^{\omega 2} \omega^4\, \text{S0}\, d\omega$$

which yields: 249 920 000

$$\alpha 2 = \frac{\lambda 2}{\sqrt{\lambda 0\, \lambda 4}} \; // \; \text{N}$$

which yields: 0.986825

A value of $\alpha 2$ near unity identifies a narrowband process.

Natural frequency

Cross-sectional area:

$$\text{A} = \text{b h}$$

which yields: 0.015

Moment of inertia:

$$\text{I} = \frac{\text{b h}^3}{12} \; // \; \text{N}$$

which yields: 3.125×10^{-6}

Mass per unit length:

$$m = A\,\rho$$

which yields: 117.75

Tributary mass, from half the beam:

$$M = m\,\frac{L}{2}$$

which yields: 353.25

Stiffness:

$$K = \frac{48\,E\,I}{L^3}$$

which yields: 138 889.

Natural frequency of vibration:

$$\omega n = \sqrt{\frac{K}{M}}$$

which yields: 19.8286

... and that corresponds to the following period in seconds:

$$Tn = \frac{2\,\pi}{\omega n}$$

which yields: 0.316874

For reference, the exact natural frequency for a simply supported beam is:

$$\omega nExact = \pi^2\,\sqrt{\frac{E\,I}{m\,L^4}}$$

which yields: 19.9736

Also for reference, the bounding periods of the load spectrum, in seconds:

$$T2 = \frac{2\,\pi}{\omega 2}\;//\;N$$

which yields: 0.785398

$$T1 = \frac{2\,\pi}{\omega 1} \;//\; N$$

which yields: 1.0472

Static response

Stanard deviation of the load:

$$\sigma F = \sqrt{\lambda 0} \;//\; N$$

which yields: 316.228

Static displacement due to one standard deviation load:

$$\frac{\sigma F}{K}$$

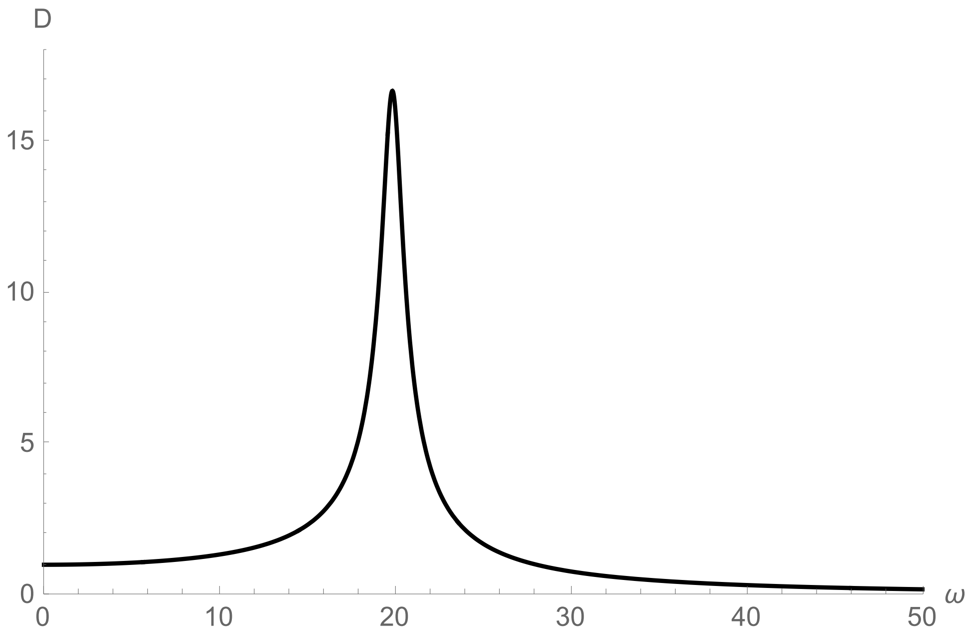
which yields: 0.00227684

Response spectrum amplitude

Dynamic amplification factor:

$$\mathbb{D} = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega n}\right)^2\right)^2 + \left(2\,\xi\,\left(\frac{\omega}{\omega n}\right)\right)^2}};$$

```
Plot[𝔻, {ω, 0, 50}, AxesLabel → {"ω", "D"}, PlotRange → {{0, 50}, {0, 18}},
 PlotStyle → Black]
```



Value of transfer function at the centre of the load spectrum:

$$\text{Dcentre} = \mathbb{D}\ /.\ \omega \to \frac{(\omega 1 + \omega 2)}{2}$$

which yields: 1.14204

That gives the following response spectrum height:

$$\text{Hcentre} = \frac{\text{Dcentre}}{\text{K}}$$

which yields: 8.22265×10^{-6}

$$\text{SU} = (\text{Hcentre})^2\ \text{S0}$$

which yields: 3.3806×10^{-6}

Response spectral moments

$$\lambda 0 = \int_{\omega 1}^{\omega 2} \text{SU}\, d\omega$$

which yields: 6.7612×10^{-6}

$$\lambda 2 = \int_{\omega 1}^{\omega 2} \omega^2\, \text{SU}\, d\omega$$

which yields: 0.000333553

$$\lambda 4 = \int_{\omega 1}^{\omega 2} \omega^4\, \text{SU}\, d\omega$$

which yields: 0.0168976

$$\alpha 2 = \frac{\lambda 2}{\sqrt{\lambda 0\, \lambda 4}} \text{ // N}$$

which yields: 0.986825

A value of $\alpha 2$ near unity identifies a narrowband process.

Standard deviation for the response and its derivative process:

$$\sigma \text{U} = \sqrt{\lambda 0} \text{ // N}$$

which yields: 0.00260023

$$\sigma \text{Udot} = \sqrt{\lambda 2} \text{ // N}$$

which yields: 0.0182634

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$$\sigma \text{Udot} = \sqrt{\lambda 2}$$

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Crossing rate

This means that the crossing rate is:

$$\text{nuPlus} = \frac{\sigma\text{Udot}}{2\,\pi\,\sigma\text{U}}\,\text{Exp}\left[-\frac{1}{2}\left(\frac{\text{r}}{\sigma\text{U}}\right)^2\right]$$

which yields: 0.342392

That means that the average time between crossings is:

$$\text{TPlus} = \frac{1}{\text{nuPlus}}$$

which yields: 2.92063

Mean amplitude of peaks

The peaks of a stationary Gaussian process have the Rayleigh distribution, which unshifted is a one-parameter distribution. When written in terms of the parameter sigma (see the document on Continuous Random Variables) the one parameter is simply σ_U, namely the standard deviation of the response process. The mean is given by:

$$\mu\text{up} = \sigma\text{U}\sqrt{\frac{\pi}{2}}$$

which yields: 0.00325891

Standard deviation of the amplitude of peaks

$$\sigma\text{up} = \sigma\text{U}\sqrt{\frac{4-\pi}{2}}$$

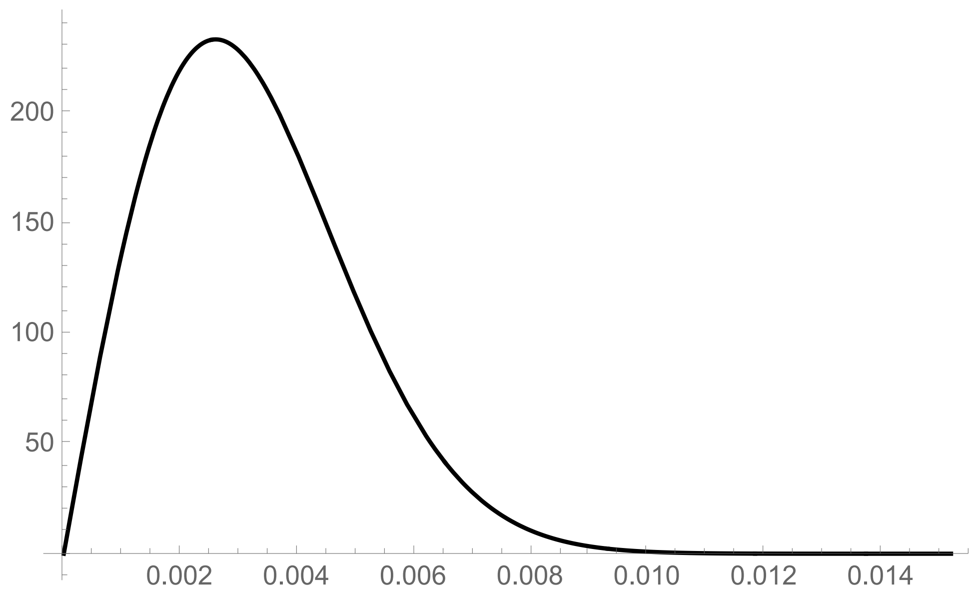
which yields: 0.00170351

Peak distribution

The PDF of the Rayleigh distribution is:

$$\text{PDFup} = \frac{\text{up}}{\sigma\text{U}^2}\,\text{Exp}\left[-\frac{\text{up}^2}{2\,\sigma\text{U}^2}\right];$$

```
Plot[PDFup, {up, 0, μup + 7 σup}, PlotStyle → Black]
```



Mean amplitude of extremes

The expected extreme response is a rather complicated expression, which contains Euler's constant γ=0.5772...:

```
T = 60 × 60;
```

$$\mu ue = \sigma U \left(\sqrt{2\,\mathrm{Log}\left[\frac{1}{2\,\pi}\ \frac{\sigma Udot}{\sigma U}\ T\right]} + \frac{EulerGamma}{\sqrt{2\,\mathrm{Log}\left[\frac{1}{2\,\pi}\ \frac{\sigma Udot}{\sigma U}\ T\right]}} \right)$$

which yields: `0.0109626`

Standard deviation of the amplitude of extremes

The expression for the standard deviation is:

$$\sigma ue = \sqrt{\frac{\pi^2\ \sigma U^2}{12\,\mathrm{Log}\left[\frac{1}{2\,\pi}\ \frac{\sigma Udot}{\sigma U}\ T\right]}}$$

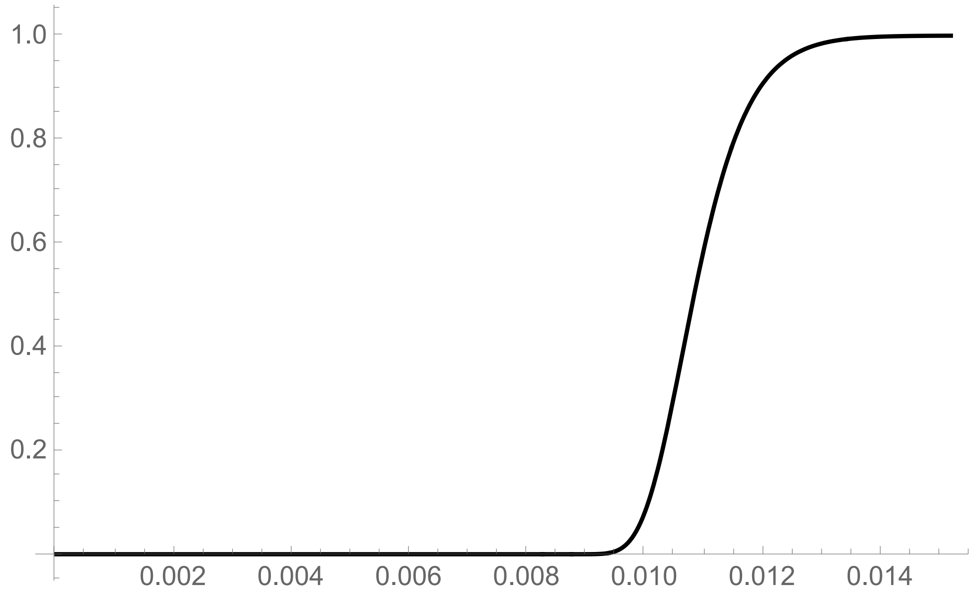
which yields: `0.00081852`

Extreme distribution

The CDF is given by:

```
Fue = Exp[- σUdot/(2 π σU) Exp[- 1/2 (ue/σU)^2] T];
```

```
Plot[Fue, {ue, 0, μup + 7 σup}, PlotStyle → Black]
```



Extreme probability

The probability of extremes greater than a threshold is the complement of the above CDF:

```
P = 1 - Fue /. ue → 3 r
```

which yields: `0.0910601`

Extreme threshold

The threshold that has 1% chance of being exceeded:

```
prob = 1 - 0.01
```

which yields: `0.99`

... of beeing exceeded in T is given by:

$$\sigma U \sqrt{2\,\mathrm{Log}\left[\frac{\frac{1}{2\,\pi}\,\frac{\sigma\mathrm{Udot}}{\sigma U}\,T}{\mathrm{Log}\left[\frac{1}{\mathrm{prob}}\right]}\right]}$$

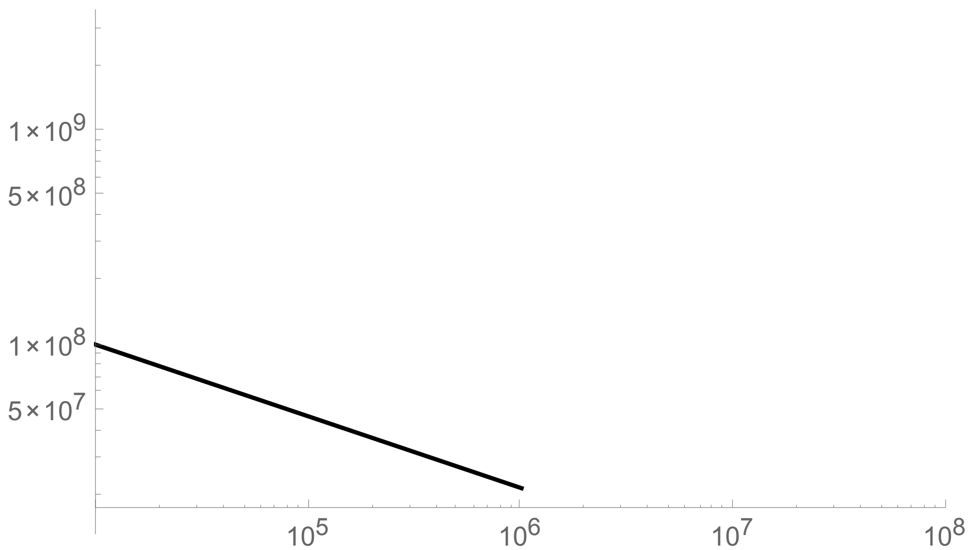
which yields: 0.0132077

Fatigue

Plot of the SN curve:

$$\mathrm{SNcurveK} = 10^{28};$$
$$\mathrm{SNcurveM} = 3.0;$$
$$\mathrm{LogLogPlot}\left[\left(\frac{\mathrm{SNcurveK}}{\mathrm{NN}}\right)^{\frac{1}{\mathrm{SNcurveM}}},\ \{\mathrm{NN},\ 1,\ 10^{6}\},\ \mathrm{PlotRange}\to\{\{10^{4},\ 10^{8}\},\ \{0,\ 400\times 10^{6}\}\},\ \mathrm{PlotStyle}\to\mathrm{Black}\right]$$

Null



Stress due to applied load, stress=(M/I)*z

$$\mathrm{stressFactor} = \frac{\left(\frac{L}{4}\right)}{I}\left(\frac{h}{2}\right)$$

which yields: 12 000.

For example, stress due to standard deviation load applied statically, in Pascal:

$$\text{stressFactor}\ \sigma F$$

which yields: 3.79473×10^6

That gives the following response spectrum height:

$$SS = (\text{stressFactor Dcentre})^2\ S0$$

which yields: 9.39056×10^{12}

$$\lambda 0 = \int_{\omega 1}^{\omega 2} SS\, d\omega$$

which yields: 1.87811×10^{13}

$$\lambda 2 = \int_{\omega 1}^{\omega 2} \omega^2\, SS\, d\omega$$

which yields: 9.26535×10^{14}

Standard deviation of the stress response, in Pascal:

$$\sqrt{\lambda 0}$$

which yields: 4.33372×10^6

Expected fatigue life, in seconds:

$$\text{expectedT} = \frac{2\,\pi\,\text{SNcurveK}\ 2^{-1.5\ \text{SNcurveM}}\ \lambda 0^{0.5\ (1-\text{SNcurveM})}\ \lambda 2^{-0.5}}{\text{Gamma}\left[1 + \frac{\text{SNcurveM}}{2}\right]}$$

which yields: 3.6539×10^6

In days, that is:

$$\frac{\text{expectedT}}{60 \times 60 \times 24}$$

which yields: 42.2905