Mean-value First-order Second-moment Method (MVFOSM)

Consider a situation in which the only information we have about the random variables is their second-moments, i.e., means, standard deviations, and correlation coefficients. Under these circumstances, analysis of functions provides the mean and standard deviation of the limit-state function, i.e., μ_g and σ_g. In MVFOSM these two values are used to construct a measure of the reliability. The measure is called the reliability index and serves as a surrogate for the failure probability.

This reliability index is accurate for linear limit-state functions but it suffers from the so-called invariance problem when the limit-state function is nonlinear. This is because if the limit-state function is linear then its mean and standard deviation are accurately computed. Conversely, if the limit-state function is nonlinear then first-order approximations are utilized.

MVFOSM reliability method is so named for three reasons: The fact that the second moments of the limit-state function are obtained by a Taylor approximation centred at the Mean Values of the random variables provides the two first letters. The next two letters indicate that the Taylor approximation is of the First Order. The last two letters state that only Second Moment information of the random variables is considered.

Second-moment Reliability Index

In MVFOSM the reliability index is defined as the ratio of the mean to the standard deviation of the limit-state function:

$$\beta = \frac{\mu_g}{\sigma_g} \tag{1}$$

The validity of β as a proxy for failure probability is understood by first imagining the PDF of the limit-state function. Unless g is a linear function of normal random variables, or some other simple case, this PDF cannot be established analytically. Nevertheless, this is a pedagogically useful thought-construct. Obviously, the PDF of g has mean μ_g and standard deviation σ_g. Furthermore, g=0 separates the failure outcomes from the safe outcomes. By multiplying Eq. (1) by σ_g it becomes clear that β is the number of standard deviations from the mean to the failure region. The more standard deviations the failure domain is away from the mean the safer. In other words, the higher reliability index the smaller is the failure probability. Conversely, small values of the reliability index indicate that the failure domain is closer to the mean, which implies a higher failure probability. In addition to this explanation, a geometric interpretation of β in the space of random variables, which will be useful in the derivation of other reliability methods, is presented later in this document.

Invariance Problem

Formulas for the computation of μ_g and σ_g are known from the analysis of functions. If the limit-station function is linear then μ_g and σ_g are exact. If it is nonlinear then first-order approximations are available:

$$\mu_g \approx g(\mathbf{M}_X) \tag{2}$$

$$\sigma_g \approx \sqrt{\nabla g(\mathbf{M}_X)^T \mathbf{\Sigma}_{XX} \nabla g(\mathbf{M}_X)} \tag{3}$$

It is the potential inaccuracy in these two equations that gives rise to the "invariance problem" of the MVFOSM method. It is called the invariance problem because the MVFOSM may provide different results for equivalent limit-state functions. Two limit-state functions are equivalent if they share the surface g=0 in the space of random variables. Consider the basic reliability problem as an example. For the limit-state function g=R-S the MVFOSM reliability index is

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S + \sigma_S^2}} \tag{4}$$

For the equivalent limit-state function g=ln(R/S) the solution is

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\ln\left(\mu_R/\mu_S\right)}{\sqrt{\sigma_R^2/\mu_R^2 - (2\rho_{RS}\sigma_R\sigma_S)/(\mu_R\mu_S) + \sigma_S^2/\mu_S^2}} \tag{5}$$

In general, Eqs. (4) and (5) yield different results. For example, for μ_R=30, μ_S=20, σ_R=5, σ_S=10, and ρ_{RS}=0.5 Eq. (4) yields β=1.15 while Eq. (5) yields β=0.92. This exemplifies the invariance problem. The root of the problem is that the linearization of the limit-state function is made at the mean. For a nonlinear limit-state functions this implies that the limit-state surface, i.e., g=0, is different for the linearized function and the actual function. It would be better to carry out the linearization somewhere at the limit-state surface g=0, which is shared by all equivalent limit-state functions. This solution gives rise to FORM and SORM, which are described in other documents.

Geometric Interpretation of the Reliability Index

To better understand the reliability index and to assist the development of FORM, a geometric interpretation of the second-moment reliability index is made. First consider a linear limit-state function of the form

$$g(\mathbf{X}) = a + \mathbf{b}^T\mathbf{X} \tag{6}$$

Furthermore, transform the function into the space of standard variables, i.e., transform **X** into a vector of variables with zero means and unit covariance matrix:

$$\mathbf{X} = \mathbf{M}_X + \mathbf{D}_X\mathbf{L}\mathbf{Y} \tag{7}$$

Substitution of Eq. (7) into Eq. (6) and naming the limit-state function in the standard space G yields

$$\begin{aligned} G(\mathbf{Y}) &= a + \mathbf{b}^T \mathbf{M}_X + \mathbf{b}^T \mathbf{D}_X \mathbf{L} \mathbf{Y} \\ &= c + \mathbf{d}^T \mathbf{Y} \end{aligned} \tag{8}$$

where c and $\mathbf{d}$ have been defined. Now carry out MVFOSM with this limit-state function:

$$\beta = \frac{\mu_G}{\sigma_G} = \frac{c}{\sqrt{\mathbf{d}^T \mathbf{d}}} = \frac{c}{\|\mathbf{d}\|} \tag{9}$$

Compare this result with the geometry formula for the distance from a point to a plane. Indeed, the distance from $\mathbf{Y}=\mathbf{0}$ (the origin in the $\mathbf{Y}$-space) to the limit-state plane $G(\mathbf{Y})=0$ is:

$$\Delta = \left| \frac{G(\mathbf{0})}{\|\nabla G\|} \right| = \left| \frac{c}{\|\mathbf{d}\|} \right| \tag{10}$$

We conclude that the reliability index β is the distance from the origin to the limit-state surface in the space of standardized random variables. This shows an important appeal of the standard space: distances can be measured. Conversely, the original random variables usually have a variety of units and a distance in that space is not a meaningful concept.

Failure Probability under the Normality Assumption

The MVFOSM method yields a reliability index, but it does not provide a failure probability. This changes in the special case where the limit-state function is linear and the random variables are normally distributed. In this case, the probability distribution of the limit-state function is also normal. Furthermore, the second-moment analysis yields the precise mean, μ_g, and standard deviation, σ_g. Consequently, the standard normal CDF is employed to obtain the failure probability:

$$p_f = P(g \leq 0) = \Phi\left(\frac{0 - \mu_g}{\sigma_g} \right) = \Phi(-\beta) \tag{11}$$

Importance Vector

Ranking of the random variables according to relative importance often provides valuable insight. The most important random variables should be subjected to particular scrutiny, while unimportant random variables may be omitted from the analysis. An importance vector from MVFOSM, i.e., a vector with components that measure the relative importance the corresponding random variable is derived by considering the linearized limit-state function at the mean point:

$$g(\mathbf{x}) \approx g(\mathbf{M_x}) + \nabla g(\mathbf{M_x})^T \cdot \left(\mathbf{x} - \mathbf{M_x} \right) \tag{12}$$

The variance of the linearized limit-state function is

$$\begin{aligned}\text{Var}\left[g(\mathbf{x})\right] &\approx \nabla g^T \cdot \mathbf{\Sigma}_{\mathbf{xx}} \cdot \nabla g \\ &= \left(\nabla g_1 \sigma_1\right)^2 + \left(\nabla g_2 \sigma_2\right)^2 + \cdots + \left(\nabla g_n \sigma_n\right)^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \nabla g_i \nabla g_j \sigma_i \sigma_j \rho_{ij}\end{aligned} \tag{13}$$

where n is the number of random variables. It is observed that the direct contribution of random variable x_i to the total variance is $\left(\nabla g_i \cdot \sigma_i\right)^2$. For this reason, the following vector is considered an importance vector in MVFOSM:

$$\boldsymbol{\omega} = -\nabla g^T \mathbf{D} \tag{14}$$

The greater absolute value of $\omega_i = \nabla g_i \cdot \sigma_i$ the greater importance of the corresponding random variable x_i. The sign of ω_i also matters:

- *Positive* ω_i means that x_i acts like a *load* variable
- *Negative* ω_i means that x_i acts like a *resistance* variable