Fixed-roller beam with point load

This is an example of an Euler-Bernoulli beam without any distributed loads. Instead the load is a mathematical singularity, namely a point load modelled by the Dirac delta function. The beam is shown in Figure 1 subjected to a vertical point load P at 2/3 of its length, L, from the fixed left end.

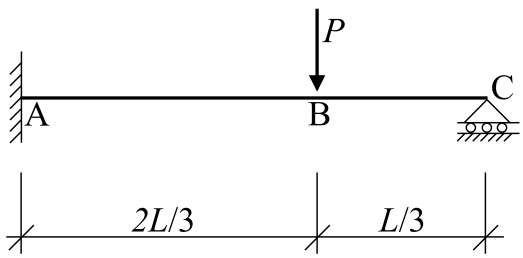


Figure 1: The beam considered in this example.

The objectives are:

- Determine analytical expressions for $w(x)$, $\theta(x)$, $M(x)$, and $V(x)$
- Determine the value of the bending moment at A and B, as well as the corresponding maximum axial stresses
- Determine the value of the shear force to the left and right of the load, as well as the corresponding maximum shear stresses
- Determine the location and value of the maximum deflection
- Determine the location(s) of the inflection points, i.e. the points with zero bending moment
- Plot w(x), θ(x), M(x), and V(x) and compute the numerical value for the previous questions when the beam has a rectangular cross-section

The following independent and dependent input variables are given, all in N and mm:

```
values = {b → 38, h → 235, E → 9500, P → 5000, L → 3000};
dependents = {I → b*h^3/12, A → b h};
```

Solution to differential equation

Governing differential equation:

$$w4 = \frac{-P\,\text{DiracDelta}\left[x - \frac{2L}{3}\right]}{E\,I};$$

Integrate once:

$$w3 = \frac{-P\,\text{HeavisideTheta}\left[x - \frac{2L}{3}\right]}{E\,I} + C1;$$

Integrate once again:

$$w2 = \frac{-P\left(x - \frac{2L}{3}\right)\text{HeavisideTheta}\left[x - \frac{2L}{3}\right]}{E\,I} + C1\,x + C2;$$

Integrate once again:

$$w1 = \frac{-P\,\frac{1}{2}\left(x - \frac{2L}{3}\right)^2\text{HeavisideTheta}\left[x - \frac{2L}{3}\right]}{E\,I} + \frac{1}{2}C1\,x^2 + C2\,x + C3;$$

Integrate once again:

$$w0 = \frac{-P\,\frac{1}{6}\left(x - \frac{2L}{3}\right)^3\text{HeavisideTheta}\left[x - \frac{2L}{3}\right]}{E\,I} + \frac{1}{6}C1\,x^3 + \frac{1}{2}C2\,x^2 + C3\,x + C4;$$

First boundary condition: w(0) = 0:

$$C4 = 0;$$

Second boundary condition: θ(0) = w' = 0:

$$C3 = 0;$$

Third boundary condition: w(L) = 0:

$$Eq1 = \frac{C2\,L^2}{2} + \frac{C1\,L^3}{6} - \frac{L^3\,P}{162\,E\,I} == 0;$$

Fourth boundary condition: M(L) = EIw'' = 0

$$Eq2 = E\,I\left(C2 + C1\,L - \frac{L\,P}{3\,E\,I}\right) == 0;$$

```
solution = Solve[{Eq1, Eq2}, {C1, C2}] // Simplify
```

which yields: $\left\{\left\{C1 \to \frac{13\,P}{27\,E\,I},\ C2 \to -\frac{4\,L\,P}{27\,E\,I}\right\}\right\}$

That solution gives the following expressions for displacement, rotation, etc:

```
w = w0 /. solution
```

which yields: $\left\{-\frac{2\,L\,P\,x^2}{27\,E\,I} + \frac{13\,P\,x^3}{162\,E\,I} - \frac{P\left(-\frac{2\,L}{3} + x\right)^3 \text{HeavisideTheta}\left[-\frac{2\,L}{3} + x\right]}{6\,E\,I}\right\}$

```
θ = w1 /. solution
```

which yields: $\left\{-\frac{4\,L\,P\,x}{27\,E\,I} + \frac{13\,P\,x^2}{54\,E\,I} - \frac{P\left(-\frac{2\,L}{3} + x\right)^2 \text{HeavisideTheta}\left[-\frac{2\,L}{3} + x\right]}{2\,E\,I}\right\}$

```
M = E I w2 /. solution
```

which yields: $\left\{E\,I\left(-\frac{4\,L\,P}{27\,E\,I} + \frac{13\,P\,x}{27\,E\,I} - \frac{P\left(-\frac{2\,L}{3} + x\right) \text{HeavisideTheta}\left[-\frac{2\,L}{3} + x\right]}{E\,I}\right)\right\}$

```
V = E I w3 /. solution
```

which yields: $\left\{E\,I\left(\frac{13\,P}{27\,E\,I} - \frac{P\,\text{HeavisideTheta}\left[-\frac{2\,L}{3} + x\right]}{E\,I}\right)\right\}$

In the following it is awkward to read some expressions because *Mathematica* does not understand that *L* is positive so that the Heaviside of a negative number is zero.

Bending moment and axial stress

Moment at A:

```
MA = M /. x → 0 // Simplify
```

which yields: $\left\{\frac{2}{27}\,L\,P\left(-2 + 9\,\text{HeavisideTheta}\left[-\frac{2\,L}{3}\right]\right)\right\}$

... which means:

$$\frac{2}{27} L\,P\,(-2)\ //\ \text{Simplify}$$

which yields: $-\frac{4\,L\,P}{27}$

Corresponding stress:

$$\text{sigmaA} = \frac{\text{MA}}{I}\frac{h}{2}$$

which yields: $\left\{\frac{h\,L\,P\left(-2 + 9\,\text{HeavisideTheta}\left[-\frac{2\,L}{3}\right]\right)}{27\,I}\right\}$

... which means:

$$\frac{h\,L\,P\,(-2)}{27\,I}\ //\ \text{Simplify}$$

which yields: $-\frac{2\,h\,L\,P}{27\,I}$

Moment at B:

$$\text{MB} = \text{M}\ /.\ x \to \frac{2\,L}{3}\ //\ \text{Simplify}$$

which yields: $\left\{\frac{14\,L\,P}{81}\right\}$

Corresponding stress:

$$\text{sigmaB} = \frac{\text{MB}}{I}\frac{h}{2}$$

which yields: $\left\{\frac{7\,h\,L\,P}{81\,I}\right\}$

... which means:

$$\frac{h\,L\,P\,(5)}{108\,I}\ //\ \text{Simplify}$$

which yields: $\frac{5\,h\,L\,P}{108\,I}$

Shear force and shear stress

Shear force to the left of the load:

```
Vleft = V /. x → 0 // Simplify
```

which yields: $\left\{\frac{13\,P}{27} - P\,\text{HeavisideTheta}\left[-\frac{2\,L}{3}\right]\right\}$

... which means:

$$\frac{13\,P}{27};$$

Corresponding shear stress:

$$\text{tauLeft} = \frac{3\,\text{Vleft}}{2\,A} \;//\; \text{Simplify}$$

which yields: $\left\{\frac{P\left(13 - 27\,\text{HeavisideTheta}\left[-\frac{2\,L}{3}\right]\right)}{18\,A}\right\}$

... which means:

$$\frac{P\,(13)}{18\,A};$$

Shear force to the right of the load:

```
Vright = V /. x → L // Simplify
```

which yields: $\left\{\frac{13\,P}{27} - P\,\text{HeavisideTheta}\left[\frac{L}{3}\right]\right\}$

... which means:

$$\frac{13\,P}{27} - P$$

which yields: $-\frac{14\,P}{27}$

$$\text{tauRight} = \frac{3\,\text{Vright}}{2\,A} \;//\; \text{Simplify}$$

which yields: $\left\{\frac{P\left(13 - 27\,\text{HeavisideTheta}\left[\frac{L}{3}\right]\right)}{18\,A}\right\}$

... which means:

$$\frac{P\,(13 - 27)}{18\,A} \;//\; \text{Simplify}$$

which yields: $-\frac{7\,P}{9\,A}$

Max deflection

Maximum deflection occurs where the rotation is zero. This gives the equation $\theta = 0$, which is a second-order equation. First try the left-hand side of the load, where the Heaviside function is zero:

$$\text{Solve}\left[-\frac{4\,L\,P\,x}{27\,E\,I} + \frac{13\,P\,x^2}{54\,E\,I} == 0,\ x\right]$$

which yields: $\left\{\{x \to 0\},\ \left\{x \to \frac{8\,L}{13}\right\}\right\}$

That is indeed the solution because it is on the left side of the load. Then there is no need to look at the right-hand side, where the Heaviside function equals unity:

$$\text{Solve}\left[-\frac{4\,L\,P\,x}{27\,E\,I} + \frac{13\,P\,x^2}{54\,E\,I} - \frac{P\left(-\frac{2\,L}{3} + x\right)^2}{2\,E\,I} == 0,\ x\right] \;//\; N$$

which yields: $\{\{x \to 0.622036\,L\},\ \{x \to 1.37796\,L\}\}$

$$\text{maxDispLocation} = \frac{8\,L}{13};$$

The value of the displacement at this location is:

$$\text{maxDisplacement} = w\,/.\,\{x \to \text{maxDispLocation}\} \;//\; \text{Simplify}$$

which yields: $\left\{\frac{4\,L^3\,P\left(-416 + \text{HeavisideTheta}\left[-\frac{2\,L}{39}\right]\right)}{177\,957\,E\,I}\right\}$

... namely:

$$\frac{4\,L^3\,P\,(-416)}{177\,957\,E\,I}\;//\;\text{Simplify}$$

which yields: $-\frac{128\,L^3\,P}{13\,689\,E\,I}$

Inflection points

Inflection points appear where the curvature is zero, i.e., where the bending moment is zero. First try on the left-hand side of the load, where the Heaviside function is zero:

$$\text{inflectionPoint} = \text{Solve}\left[E\,I\left(-\frac{4\,L\,P}{27\,E\,I}+\frac{13\,P\,x}{27\,E\,I}\right) == 0,\ x\right]$$

which yields: $\left\{\left\{x \to \frac{4\,L}{13}\right\}\right\}$

Then try on the right-hand side, where the Heaviside function equals unity:

$$\text{Solve}\left[E\,I\left(-\frac{4\,L\,P}{27\,E\,I}+\frac{13\,P\,x}{27\,E\,I}+\frac{P\left(\frac{2\,L}{3}-x\right)}{E\,I}\right) == 0,\ x\right]$$

which yields: $\{\{x \to L\}\}$

Numerical results

```
MA /. values // N
```

which yields: $\{-2.22222\times 10^6\}$

```
sigmaA /. values // N
```

which yields: $\left\{-\frac{2.61111\times 10^8}{I}\right\}$

```
MB /. values // N
```

which yields: $\{2.59259\times 10^6\}$

```
sigmaB /. dependents /. values // N
```

which yields: {7.41253}

```
Vleft /. values // N
```

which yields: {2407.41}

```
tauLeft /. dependents /. values // N
```

which yields: {0.40438}

```
Vright /. values // N
```

which yields: {-2592.59}

```
tauRight /. dependents /. values // N
```

which yields: {-0.435486}

```
maxDispLocation /. values // N
```

which yields: 1846.15

```
maxDisplacement /. dependents /. values // N
```

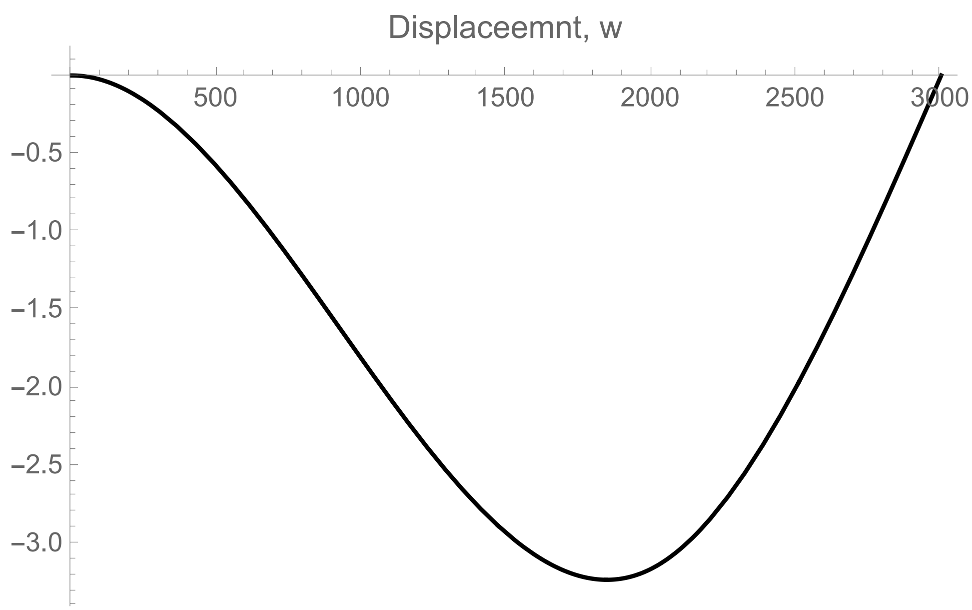
which yields: {-3.23327}

```
inflectionPoint /. values // N
```

which yields: {{x → 923.077}}

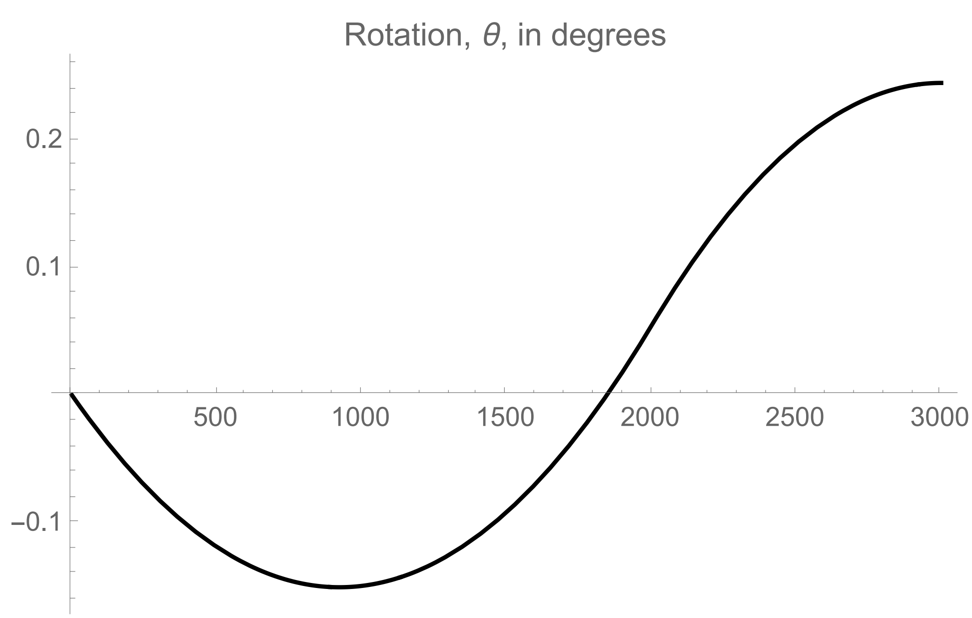
Displacement plot

```
Plot[w /. dependents /. values, {x, 0, L /. values},
 PlotLabel → "Displaceemnt, w", PlotStyle → Black]
```



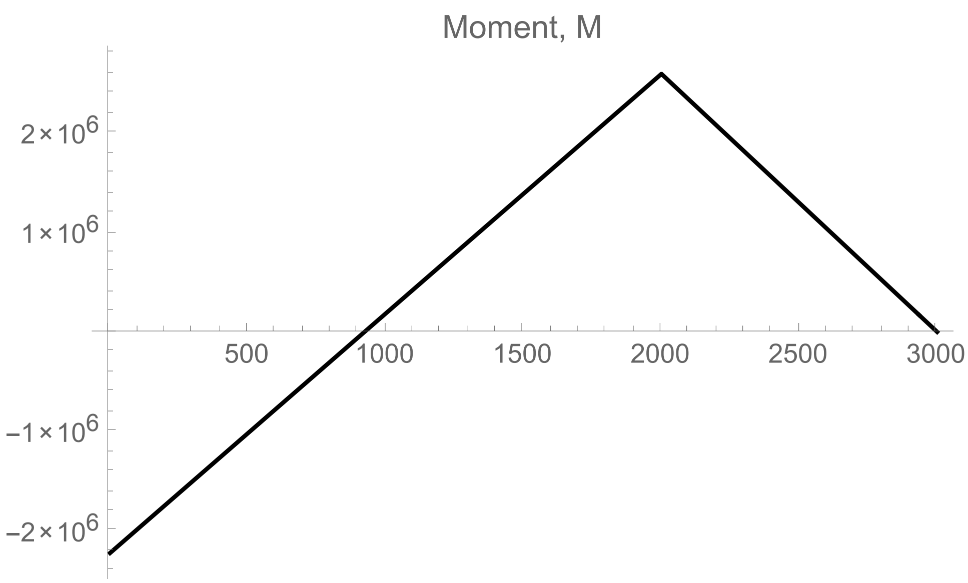
Rotation plot

$$\text{Plot}\left[\frac{\theta \text{ /. dependents /. values}}{\pi} 180, \{x, 0, L \text{ /. values}\},\right.$$
$$\left.\text{PlotLabel} \to \text{"Rotation, } \theta \text{, in degrees"}, \text{PlotStyle} \to \text{Black}\right]$$



Bending moment plot

```
Plot[M /. dependents /. values, {x, 0, L /. values},
 PlotLabel → "Moment, M", PlotStyle → Black]
```



Shear force plot

```
Plot[V /. dependents /. values, {x, 0, L /. values},
 PlotLabel → "Shear force, V", PlotStyle → Black]
```