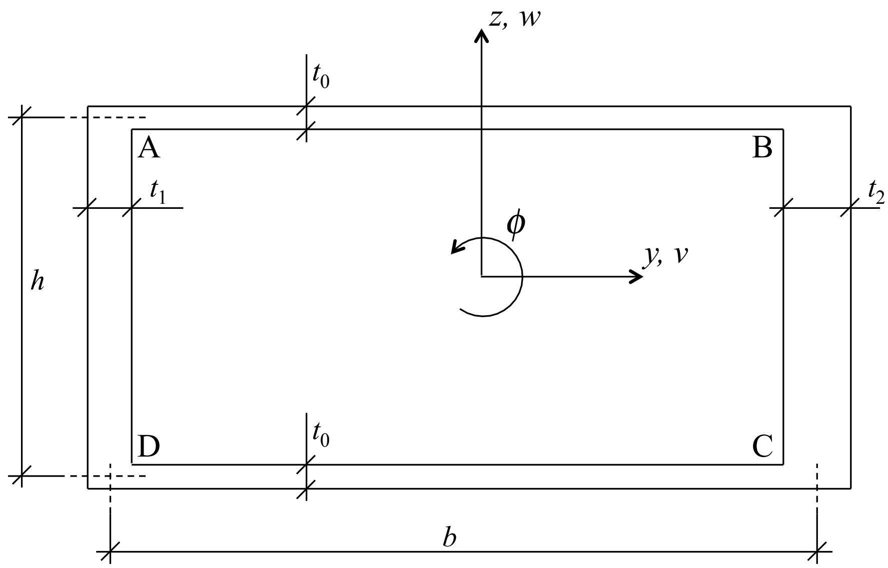
Closed rectangular cross-section

The cross-section shown in the figure below is considered with the objective of calculating all cross-section constants and determine stresses due to given stress resultants. The cross-section is "thin-walled" and the dimensions *b* and *h* are given to the centre-lines of the cross-section parts.



Input values

Dimensions:

```
b = 500 mm ;
h = 200 mm ;
t0 = 10 mm ;
t1 = 15 mm ;
t2 = 20 mm ;
```

Stress resultants:

```
N = 50 kN ;
My = 50 m kN ;
Mz = 50 m kN ;
Vz = 50 kN ;
Vy = 50 kN ;
TStV = 0.5 m kN ;
B = 0.5 kN * m^2 ;
```

Axial stress due to N

The cross-section area is:

```
A = 2 b t0 + h t1 + h t2
```

which yields: $17\,000\ \mathrm{mm}^2$

The axial stress is distributed uniformly over the cross-section with value

$$\sigma N = \frac{N}{A};$$

```
UnitConvert[σN, "N/mm2"] // N
```

which yields: $2.94118\ \mathrm{N/mm}^2$

Axial stress due to My

The moment of inertia about the y-axis is

$$Iy = \frac{t1\,h^3}{12} + \frac{t2\,h^3}{12} + 2\,b\,t0\,\left(\frac{h}{2}\right)^2 \;//\; N$$

which yields: $1.23333 \times 10^8\ \mathrm{mm}^4$

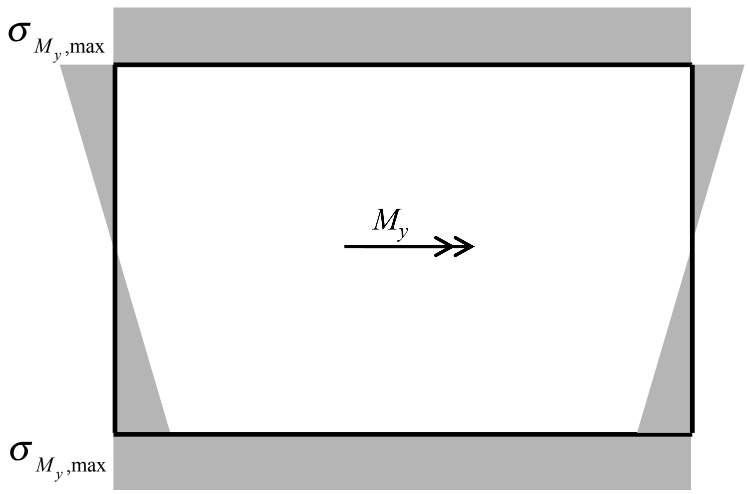
The maximum stress at the mid-line of the outer walls is the same at both walls because of symmetry, with value

$$\sigma MyMax = \frac{My}{Iy}\left(\frac{h}{2}\right);$$

```
UnitConvert[σMyMax, "N/mm2"]
```

which yields: $40.5405\ \mathrm{N/mm^2}$

Calculating that stress value we could have used $\frac{h+\frac{t_0}{2}}{2}$ instead of $\frac{h}{2}$ to get the stress in the very outermost fibre, but that would not make a big difference. That axial stress is distributed as follows over the cross-section:



Axial stress due to Mz

Here we first need to compute the location of the neutral axis along the y-axis. Relative to the mid-line of the left sidewall it is:

$$y0 = \frac{2\ b\ t0\ \frac{b}{2} + h\ t2\ b}{A} \text{ // N}$$

which yields: $264.706\ \mathrm{mm}$

The relevant moment of inertia is

$$Iz = h\ t1\ y0^2 + h\ t2\ (b - y0)^2 + 2\left(\frac{t0\ b^3}{12} + t0\ b\left(\frac{b}{2} - y0\right)^2\right)$$

which yields: $6.42157 \times 10^8\ \mathrm{mm^4}$

Stress:

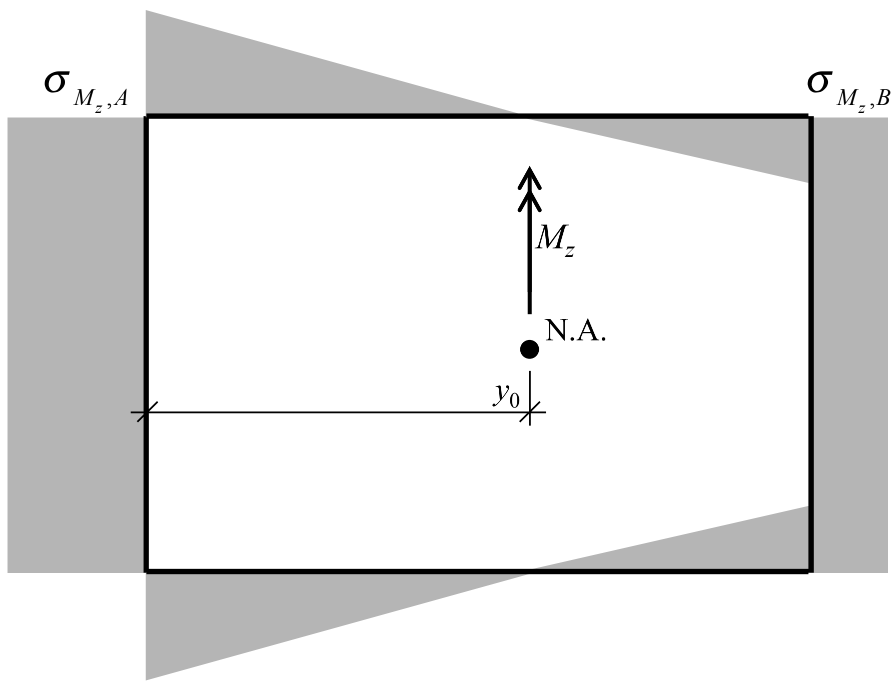
```
σMzA = Mz/Iz y0;
UnitConvert[σMzA, "N/mm2"]
```

which yields: 20.6107 N/mm^2

```
σMzB = Mz/Iz (b - y0);
UnitConvert[σMzB, "N/mm2"]
```

which yields: 18.3206 N/mm^2

The distribution of this axial stress over the cross-section looks like this:



Shear stress due to Vz

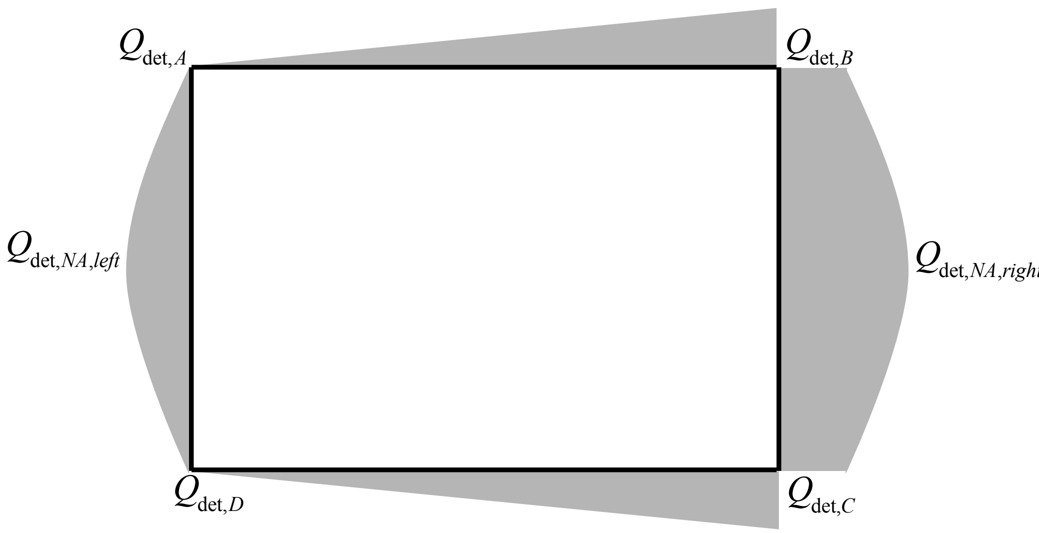
This is a closed cross-section, hence the shear flow is statically indeterminate. That means that the shear flow and shear stress cannot be determined by equilibrium alone. To solve the problem we essentially use the same approach as the Flexibility Method in structural analysis. Hence, we select to make a cut at A and use a compatibility equation to glue the cross-section back together. The unknown shear flow at A is the redundant, named q_0.

APPROACH 1

In this approach we integrate the first moment of area, Q, of the statically determinate cross-section and determine Q at the cut by the formula

$$Q0 = -\frac{\int \frac{Qdet}{t}\, ds}{\int \frac{1}{t}\, ds}$$

The statically determinate Q diagram is shown in this figure, and the values are computed in the folowing, starting at A and going clockwise around the cross-section:



$$QdetB = t0\ b\ \frac{h}{2}$$

which yields: $500\,000\ \text{mm}^3$

$$QdetNAright = QdetB + t2\ \frac{h}{2}\ \frac{h}{4}$$

which yields: $600\,000\ \text{mm}^3$

$$QdetC = QdetNAright - t2\ \frac{h}{2}\ \frac{h}{4}$$

which yields: $500\,000\ \text{mm}^3$

$$QdetD = QdetC - t0\, b\, \frac{h}{2}$$

which yields: $0\ \text{mm}^3$

$$QdetNAleft = QdetD - t1\, \frac{h}{2}\, \frac{h}{4}$$

which yields: $-75\,000\ \text{mm}^3$

$$QdetA = QdetNAleft + t1\, \frac{h}{2}\, \frac{h}{4}$$

which yields: $0\ \text{mm}^3$

Integration around the cell is done with the followign formula, where we notice that no minus signs are needed in the integral because the Qdet values already have the correct signs:

$$Q0method1 = -\frac{1}{2\,\frac{b}{t0} + \frac{h}{t1} + \frac{h}{t2}} \left(\frac{1}{2}\,\frac{QdetB}{t0}\,b + \frac{QdetB}{t2}\,h + \frac{2}{3}\,\frac{(QdetNAright - QdetB)}{t2}\,h + \frac{1}{2}\,\frac{QdetC}{t0}\,b + \frac{2}{3}\,\frac{QdetNAleft}{t1}\,h\right) // N$$

which yields: $-243\,243.\ \text{mm}^3$

Having the value of Q_0 at the cut means that the shear flow and shear stress can be computed, as is shown below after getting the same result with Approach 2.

APPROACH 2

Here the value of the shear flow at the cut is determined from the integral

$$q0 = \frac{Vmax}{Iy} \int g\, z\, t\, ds$$

That formula implies that the value of the first moment of area at the cut, Q_0, is:

$$Q0 = \int g\, z\, t\, ds$$

The function $g(s)$ is established by first adding all contributions of ds/t around the cross-section:

$$gSum = 2\,\frac{b}{t0} + \frac{h}{t1} + \frac{h}{t2}\;//\;N$$

which yields: 123.333

Then the normalized g-values are calculated at the different locations around the cross-section as follows:

$$gB = \frac{b}{t0}\,\frac{1}{gSum}$$

which yields: 0.405405

$$gC = gB + \frac{h}{t2}\,\frac{1}{gSum}$$

which yields: 0.486486

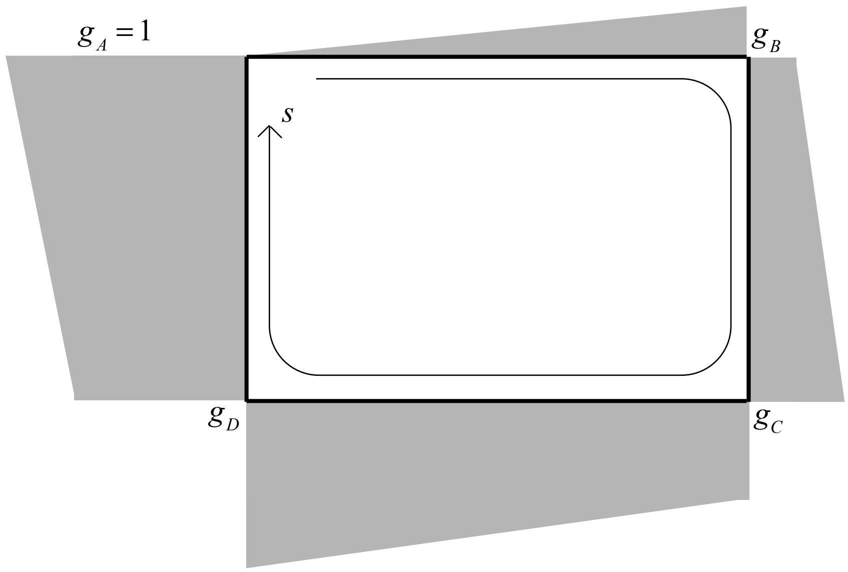
$$gD = gC + \frac{b}{t0}\,\frac{1}{gSum}$$

which yields: 0.891892

$$gA = gD + \frac{h}{t1}\,\frac{1}{gSum}$$

which yields: 1.

That $g(s)$ diagram is shown here:



To ease the evaluation of the integral for Q_0, formulas for $g(s)$ are established for each cross-section part:

$$gAB = \frac{gB}{b}\, s;$$

$$gBC = gB + \frac{gC - gB}{h}\, s;$$

$$gCD = gC + \frac{gD - gC}{b}\, s;$$

$$gDA = gD + \frac{gAend - gD}{h}\, s;$$

It is also useful to establish formulas for z for each cross-section part:

$$zAB = \frac{h}{2};$$

$$zBC = \frac{h}{2} - s;$$

$$zCD = -\frac{h}{2};$$

$$zDA = -\frac{h}{2} + s;$$

Then the full integral is:

```
Q0method2 = ∫_0^b gAB zAB t0 ⅆs + ∫_0^h gBC zBC t2 ⅆs + ∫_0^b gCD zCD t0 ⅆs +
  ∫_0^h gDA zDA t1 ⅆs
```

which yields: (-293 243. + 50 000 gAend) mm^3

From that reference value Q_0, which thankfully came out equal in both methods, the shear stress at different locations is:

```
τAtop = Vz/(Iy t0) Q0method1;
UnitConvert[τAtop, "N/mm2"]
```

which yields: -9.86121 N/mm^2

```
τBtop = Vz/(Iy t0) (Q0method1 + QdetB);
UnitConvert[τBtop, "N/mm2"]
```

which yields: 10.4091 N/mm^2

```
τBside = Vz/(Iy t2) (Q0method1 + QdetB);
UnitConvert[τBside, "N/mm2"]
```

which yields: 5.20453 N/mm^2

```
τNAright = Vz/(Iy t2) (Q0method1 + QdetNAright);
UnitConvert[τNAright, "N/mm2"]
```

which yields: 7.23156 N/mm^2

```
τCside = Vz/(Iy t2) (Q0method1 + QdetC);
UnitConvert[τCside, "N/mm2"]
```

which yields: 5.20453 N/mm^2

```
τCbottom = Vz/(Iy t0) (Q0method1 + QdetC);
UnitConvert[τCbottom, "N/mm2"]
```

which yields: $10.4091\ \mathrm{N/mm^2}$

```
τDbottom = Vz/(Iy t0) (Q0method1 + QdetD);
UnitConvert[τDbottom, "N/mm2"]
```

which yields: $-9.86121\ \mathrm{N/mm^2}$

```
τDside = Vz/(Iy t1) (Q0method1 + QdetD);
UnitConvert[τDside, "N/mm2"]
```

which yields: $-6.57414\ \mathrm{N/mm^2}$

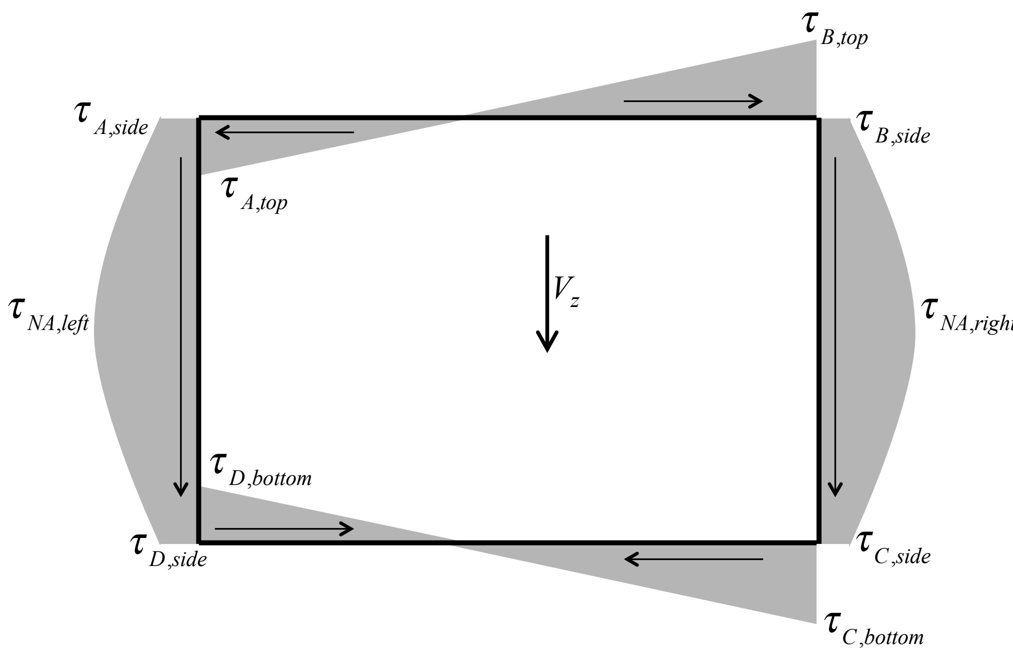
```
τNAleft = Vz/(Iy t1) (Q0method1 + QdetNAleft);
UnitConvert[τNAleft, "N/mm2"]
```

which yields: $-8.60117\ \mathrm{N/mm^2}$

```
τAside = Vz/(Iy t1) (Q0method1);
UnitConvert[τAside, "N/mm2"]
```

which yields: $-6.57414\ \mathrm{N/mm^2}$

Those shear stresses are distributed over the cross-section as follows:



Shear centre by equilibrium

Force in left sidewall:

$$\text{leftForce} = \text{Abs}\left[\tau\text{Dside}\ \text{t1}\ \text{h} + \frac{2}{3}\left(\tau\text{NAleft} - \tau\text{Dside}\right)\ \text{t1}\ \text{h}\right]$$

which yields: 23.7765 kN

Force in right sidewall:

$$\text{rightForce} = \tau\text{Bside}\ \text{t2}\ \text{h} + \frac{2}{3}\left(\tau\text{NAright} - \tau\text{Bside}\right)\ \text{t2}\ \text{h}$$

which yields: 26.2235 kN

For the record, check that the two sidewall forces add up to the total shear force Vz:

$$\text{leftForce} + \text{rightForce}$$

which yields: 50. kN

By the way, notice that the percentage shear force in each sidewall is NOT proportional to the respective thicknesses:

```
leftForce / (leftForce + rightForce) 100
```

which yields: 47.553

... versus:

```
t1 / (t1 + t2) 100 // N
```

which yields: 42.8571

Length of right-hand-side stress triangles in flanges shown above:

```
x = τBtop / (Abs[τAtop] + τBtop) b
```

which yields: 256.757 mm

Force in each right-hand-side triangle in flanges:

```
topRightForce = 1/2 x τBtop t0
```

which yields: 13.363 kN

Force in each left-hand-side triangle in flanges:

```
topLeftForce = 1/2 (b - x) Abs[τAtop] t0
```

which yields: 11.9934 kN

The location of shear centre is now calculated by equilibrium as the distance from the mid-line of the left wall, denoted y_{sc}:

```
equation = leftForce distance + topLeftForce h ==
   rightForce (b - distance) + topRightForce h;
solution = Solve[equation, distance];
ysc = (distance /. solution)[[1]];
UnitConvert[ysc, "mm"]
```

which yields: 267.714 mm

Shear stress due to Vy

The cross-section is symmetric about the y-axis, hence the shear stress is zero at the midpoint between B and C, and at the midpoint between A and D. As a result the calculation of shear stress due to Vy is much simpler than that of Vz above. The shear stress values are as following, with symbols explained in the figure:

$$QVyA = t1 \frac{h}{2} y0;$$
$$\tau VyAside = \frac{Vy}{Iz\, t1} QVyA;$$
UnitConvert[τVyAside, "N/mm2"]

which yields: $2.06107\ \mathrm{N/mm^2}$

$$\tau VyAtop = \frac{Vy}{Iz\, t0} QVyA;$$
UnitConvert[τVyAtop, "N/mm2"]

which yields: $3.0916\ \mathrm{N/mm^2}$

$$QVyNA = QVyA + t0\, y0 \frac{y0}{2};$$
$$\tau VyNA = \frac{Vy}{Iz\, t0} QVyNA;$$
UnitConvert[τVyNA, "N/mm2"]

which yields: $5.81949\ \mathrm{N/mm^2}$

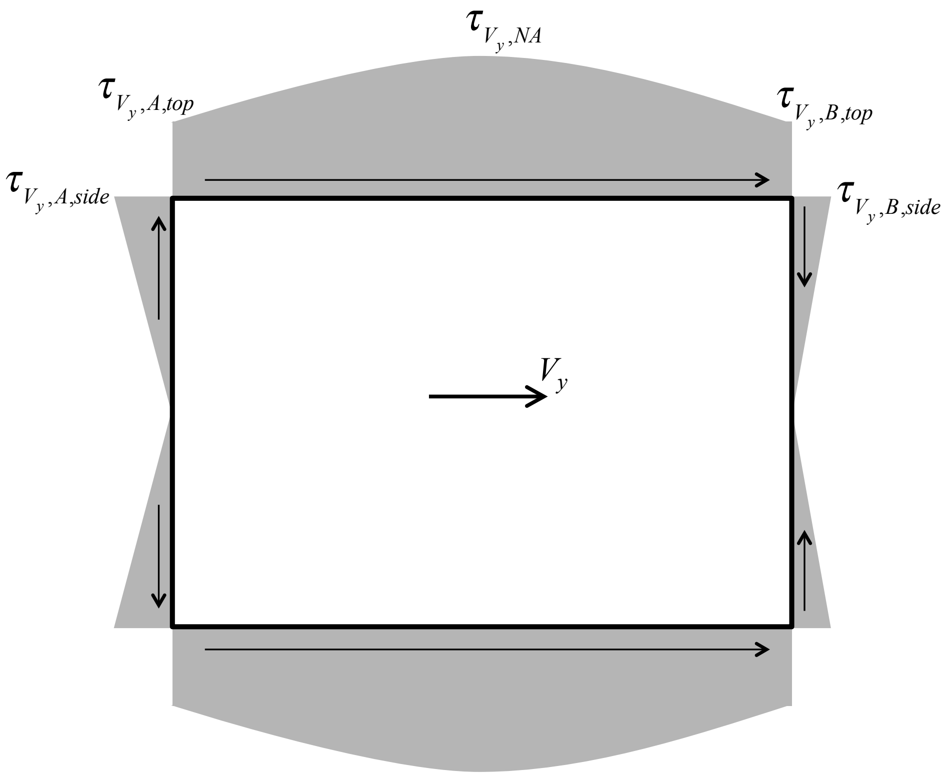
$$QVyB = QVyNA - t0\,(b - y0) \frac{(b - y0)}{2};$$
$$\tau VyBtop = \frac{Vy}{Iz\, t0} QVyB;$$
UnitConvert[τVyBtop, "N/mm2"]

which yields: $3.66412\ \mathrm{N/mm^2}$

$$\tau VyBside = \frac{Vy}{Iz\, t2} QVyB;$$
UnitConvert[τVyBside, "N/mm2"]

which yields: $1.83206\ \mathrm{N/mm^2}$

This shear stress is distributed as follows over the cross-section:



Shear stress due to $T_{St.V.}$

Area within the midline of the cross-section parts:

```
Am = b h
```

which yields: 100 000 mm^2

Cross-section constant for St. Venant torsion:

$$J = \frac{4\,Am^2}{\frac{2\,b}{t0} + \frac{h}{t1} + \frac{h}{t2}} \quad // \; N$$

which yields: 3.24324×10^8 mm^4

Shear flow due to St. Venant torsion:

```
K = TStV / (2 Am) // N
```

which yields: 2.5 kN/m

Shear stress in upper/lower cross-section wall:

```
τStVtop = K / t0;
UnitConvert[τStVtop, "N/mm2"]
```

which yields: $0.25\ \mathrm{N/mm^2}$

Shear stress in left wall:

```
τStVleft = K / t1;
UnitConvert[τStVleft, "N/mm2"]
```

which yields: $0.166667\ \mathrm{N/mm^2}$

Shear stress in right wall:

```
τStVright = K / t2;
UnitConvert[τStVright, "N/mm2"]
```

which yields: $0.125\ \mathrm{N/mm^2}$

These shear stresses flow around in the cross-section, being constant over the thickness of each wall.

Axial stress due to B

Shear radius for upper/lower flange:

```
h0 = J / (2 t0 Am)
```

which yields: 162.162 mm

Shear radius to left web:

$$h1 = \frac{J}{2\, t1\, Am}$$

which yields: 108.108 mm

Shear radius to right web:

$$h2 = \frac{J}{2\, t2\, Am}$$

which yields: 81.0811 mm

Select the centroid as the trial point Q, and do “radar sweep” clockwise starting the “beam” pointing horizontally to the left because we know the omega diagram is zero there due to symmetry:

$$\Omega QA = (y0 - h1)\, \frac{h}{2}$$

which yields: $15\,659.8\ \text{mm}^2$

$$\Omega QB = \Omega QA + \left(\frac{h}{2} - h0\right) b$$

which yields: $-15\,421.3\ \text{mm}^2$

$$\Omega QC = \Omega QB + ((b - y0) - h2)\, h$$

which yields: $15\,421.3\ \text{mm}^2$

$$\Omega QD = \Omega QC + \left(\frac{h}{2} - h0\right) b$$

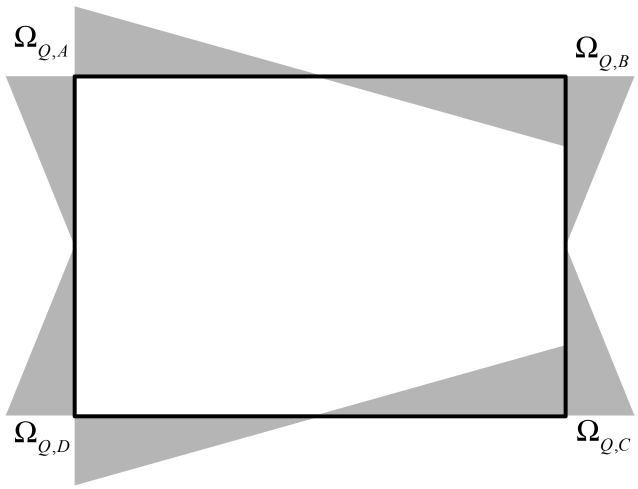
which yields: $-15\,659.8\ \text{mm}^2$

Check that we get back to zero at the point where we started the sweep:

$$\Omega QD + (y0 - h1)\, \frac{h}{2}$$

which yields: $-3.63798 \times 10^{-12}\ \text{mm}^2$

That trial diagram looks like this:



Because the cross-section is symmetric about the y-axis the normalizing constant, *C*, is zero. The integral of z Ω_Q is:

$$\text{OmegaZIntegral} = 2\left(\int_0^{\frac{h}{2}} s\left(\frac{\Omega QA}{\frac{h}{2}}\,s\right) t1\,ds + \int_0^{b} \frac{h}{2}\left(\Omega QA + \left(\frac{h}{2} - h0\right) s\right) t0\,ds + \int_0^{\frac{h}{2}} s\left(\frac{\Omega QB}{\frac{h}{2}}\,s\right) t2\,ds\right)$$

which yields: $-3.70959\times 10^{8}\ \text{mm}^{5}$

Shear centre coordinate, relative to the centroid is

$$\text{ysc} = -\frac{\text{OmegaZIntegral}}{\text{Iy}}$$

which yields: 3.00778 mm

That compares well with the shear centre coordinate obtained earlier by equilibrium:

$$\text{y0} + \text{ysc}$$

which yields: 267.714 mm

The final omega diagram is $\Omega = \Omega_Q + (y_{sc} - y_Q)\,z$, where $y_Q = 0$, i.e., it is obtained by adding the "($y_{sc}\,z$) diagram" to the trial omega diagram. For that purspoe, the values of the "($y_{sc}\,z$) diagram" is calculated at key locations:

$$\text{yscZUpperFlange} = \frac{h}{2}\,\text{ysc}$$

which yields: $300.778\ \text{mm}^2$

$$\text{yscZLowerFlange} = -\frac{h}{2}\,\text{ysc}$$

which yields: $-300.778\ \text{mm}^2$

The final omega diagram is then:

$$\Omega A = \Omega QA + \text{yscZUpperFlange}$$

which yields: $15\,960.6\ \text{mm}^2$

$$\Omega B = \Omega QB + \text{yscZUpperFlange}$$

which yields: $-15\,120.5\ \text{mm}^2$

$$\Omega C = \Omega QC + \text{yscZLowerFlange}$$

which yields: $15\,120.5\ \text{mm}^2$

$$\Omega D = \Omega QD + \text{yscZLowerFlange}$$

which yields: $-15\,960.6\ \text{mm}^2$

The final omega diagram looks very similar to the trial diagram shown in the figure above. Using "quick integration" formulas the cross-sectional constant for warping torsion is

$$\text{Cw} = 2\left(\frac{1}{3}\,\Omega A^2\,\frac{h}{2}\,\text{t1} + \frac{1}{3}\,\Omega A^2\,\text{b}\,\text{t0} + \frac{1}{3}\,\Omega C^2\,\text{b}\,\text{t0} - \frac{2}{6}\,\Omega C\,\Omega A\,\text{b}\,\text{t0} + \frac{1}{3}\,\Omega C^2\,\frac{h}{2}\,\text{t2}\right)$$

which yields: $1.36637\times 10^{12}\ \text{mm}^6$

Finally, the axial stresses due to warping torsion have the following values, distributed proportional to the amplitudes of the omega diagram:

$$\sigma A = \frac{B}{Cw}\,\Omega A;$$

```
UnitConvert[σA, "N/mm2"]
```

which yields: 5.84049 N/mm^2

$$\sigma B = \frac{B}{Cw}\,\Omega B;$$

```
UnitConvert[σB, "N/mm2"]
```

which yields: −5.53309 N/mm^2

$$\sigma C = \frac{B}{Cw}\,\Omega C;$$

```
UnitConvert[σC, "N/mm2"]
```

which yields: 5.53309 N/mm^2

$$\sigma D = \frac{B}{Cw}\,\Omega D;$$

```
UnitConvert[σD, "N/mm2"]
```

which yields: −5.84049 N/mm^2