3D Elasticity Theory

Many structural analysis problems are analysed using the theory of elasticity in which Hooke's law is used to enforce proportionality between stress and strain at any deformation level. Beams, plates, and shells are examples of such problems. Here the general equations for equilibrium, material law, and kinematic compatibility of a 3D continuum are established.

Equilibrium

Here we are not concerned with externally applied loads or surface forces but rather equilibrium between the forces within a material particle. This includes stresses and potential forces acting on the volume, such as gravity. Figure 1 shows the stresses acting on an infinitesimally small material volume. The figure is hard to read because it includes the changes in stress-values from one end of the cube to the opposite side. For example, the axial stress σ_{xx} changes by $d\sigma_{xx}$ from one side to the other. The volume forces are not included in the figure but are denoted f_x, f_y, and f_z with index indicating the direction of the force. Equilibrium in the x-direction yields

$$d\sigma_{xx} \cdot dy \cdot dz + d\sigma_{yx} \cdot dx \cdot dz + d\sigma_{zx} \cdot dx \cdot dy + f_x \cdot dx \cdot dy \cdot dz = 0 \tag{1}$$

Dividing through by ($dx\ dy\ dz$) yields:

$$\frac{d\sigma_{xx}}{dx} + \frac{d\sigma_{yx}}{dy} + \frac{d\sigma_{zx}}{dz} + f_x = 0 \tag{2}$$

Repeating the exercise in all three axis-directions produces the equilibrium equations that all material particles that are in equilibrium must satisfy:

$$\sigma_{ij,i} + f_j = 0 \tag{3}$$

Moment equilibrium of the infinitesimal cube yields the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji} \tag{4}$$

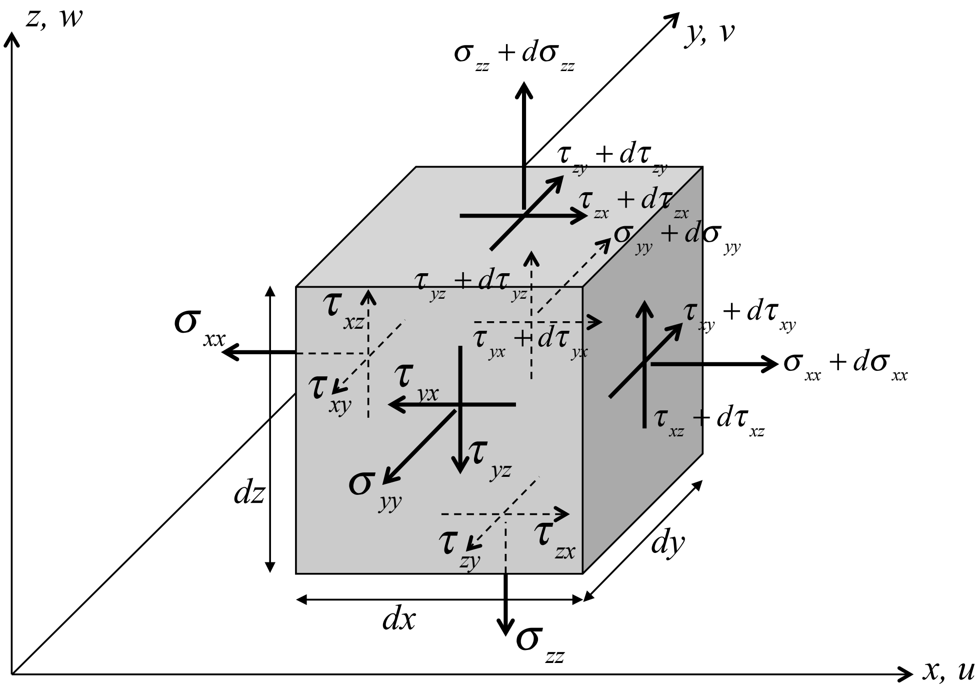

Figure 1: Stresses on a solid cube.

Kinematic Compatibility

Equations are here sought to relate strains with displacements. As shown in Figure 1 the displacements in the axis directions *x, y, z* are *u v w*, respectively. Expressions for longitudinal strains are obtained by studying an infinitesimal material cube. First consider the *x*-direction. The *x*-direction displacement at *x* is *u*. The *x*-direction displacement at *x*+d*x* is $u+(\partial u/\partial x)dx$. Defining strain as change in length divided by original length yields

$$\varepsilon_{xx} = \frac{\left(\frac{\partial u}{\partial x}\right) \cdot dx}{dx} = \frac{\partial u}{\partial x} \tag{5}$$

Repeating the consideration for the other axis directions yields

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \tag{6}$$

Now to the shear strains, starting with γ_{xy} visualized in Figure 2 and defining the change in angle between originally orthogonal lines. The change in angle has two contributions:

$$\gamma_{xy} = \gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\left(\frac{\partial v}{\partial x}\right)\cdot dx}{dx} + \frac{\left(\frac{\partial u}{\partial y}\right)\cdot dy}{dy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{7}$$

Here it is understood that the engineering shear strain γ_{xy} is twice the corresponding coordinate strains. Repeating the consideration for the other two coordinate planes yields

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned} \tag{8}$$

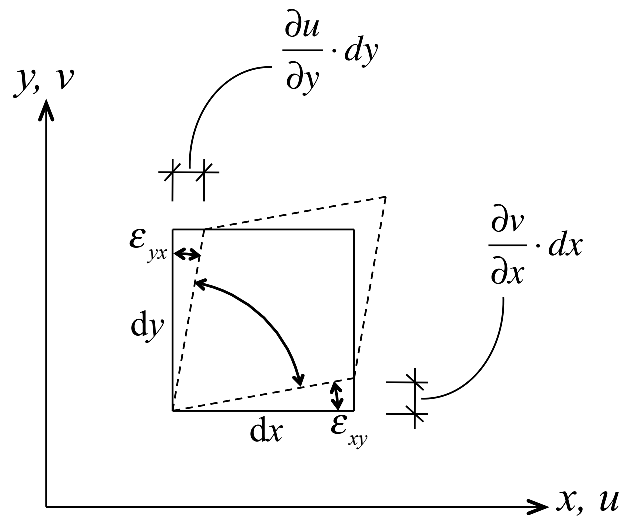

Figure 2: Shear strains.

The kinematic compatibility equations for both axial and shear strains are summarized by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{9}$$

By means of Voight notation the kinematic equations can be written in vector notation, with ∇ defined as a matrix differential operator:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ v_{,z} + w_{,y} \\ u_{,z} + w_{,x} \end{bmatrix} = \nabla \mathbf{u} \tag{10}$$

Because the compatibility equations contain six strain components and only three displacement components, only certain strain patterns are physically possible. For that reason the strains-displacement equations are sometimes combined into "compatibility equations" that give conditions for valid deformation patterns. This is done below for 2D elasticity theory. Sometimes the terms compatibility equation and compatibility condition are maintained even when adding material law and equilibrium equations. Such equations are, together with boundary conditions, sufficient to determine the solution to specific problems.

Material Law

The theory of elasticity is founded on the assumption of a homogeneous isotropic linear elastic material. For a material particle, the relationship between a uniaxial stress and the corresponding uniaxial strain is give by the modulus of elasticity, sometimes called Young's modulus, E, formulated in Hooke's law:

$$\sigma = E \cdot \varepsilon \tag{11}$$

The strain in the transversal direction is defined by Poisson's ratio, ν:

$$\nu \equiv \frac{\varepsilon_t}{\varepsilon} \qquad \Rightarrow \qquad \varepsilon_t = \nu \cdot \varepsilon \qquad \Rightarrow \qquad \varepsilon_t = -\nu \cdot \frac{\sigma}{E} \tag{12}$$

where ε_t is the transversal strain. Strain expressions that account for transversal strains in the orthogonal directions yield the three-dimensional version of Hooke's law:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{zz}}{E} \\ \varepsilon_{zz} &= \frac{\sigma_{zz}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} \end{aligned} \tag{13}$$

There are only two independent parameters in the general Hooke's law. However, a special material constant named the shear modulus, G, which is related to E and ν, defines the relationship between shear stresses and shear strains:

$$\tau_{ij} = G \cdot \gamma_{ij} \quad , \quad i \neq j \tag{14}$$

To determine the relationship between G, E and ν, consider an infinitesimally small two-dimensional material particle subjected to pure shear τ. Mohr's circle for this case is centred at the origin with radius τ. Consequently, the principal stresses are $-\tau$ and τ with axes at 45^o. The deformation of the particle is shown in Figure 3.

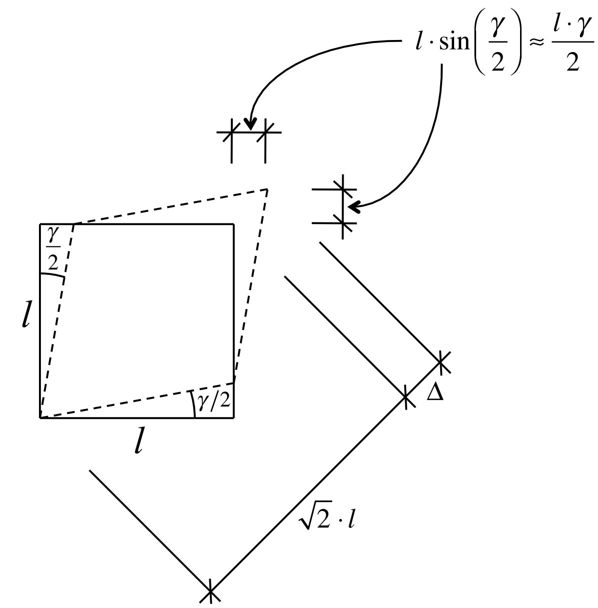

Figure 3: Derivation of the expression *G*.

The quantity Δ can be expressed in two ways. In the pure shear state:

$$\Delta = \sqrt{2 \cdot \left(\frac{l \cdot \gamma}{2}\right)^2} = \frac{l \cdot \gamma}{\sqrt{2}} \tag{15}$$

In the rotated state of pure axial stress:

$$\Delta = \varepsilon \cdot \left(\sqrt{2} \cdot l\right) = \left(\frac{\tau}{E} - \nu \cdot \frac{(-\tau)}{E}\right) \cdot \left(\sqrt{2} \cdot l\right) \tag{16}$$

Equating the two expressions for Δ yields:

$$\tau = \underbrace{\left(\frac{E}{2 \cdot (1+\nu)}\right)}_{\equiv G} \cdot \gamma \tag{17}$$

Hence, together with Eq. (13) the following equations complete the general Hooke's law:

$$\tau_{xy} = G \cdot \gamma_{xy} \quad , \quad \tau_{yz} = G \cdot \gamma_{yz} \quad , \quad \tau_{zx} = G \cdot \gamma_{zx} \tag{18}$$

In Voight notation it reads $\varepsilon_i = C_{ij}^{-1} \sigma_j$ or $\boldsymbol{\varepsilon} = \mathbf{C}^{-1} \boldsymbol{\sigma}$:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \cdot \left[\begin{array}{ccc|ccc} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{array}\right] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \tag{19}$$

Or, inversely $\sigma_i = C_{ij}\varepsilon_j$ or $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \left[\begin{array}{ccc|ccc} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{array} \right] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{20}$$

In index notation with the original strain and stress tensors, Hooke's law is written

$$\sigma_{ij} = \lambda \cdot \varepsilon_{kk} \cdot \delta_{ij} + 2 \cdot \mu \cdot \varepsilon_{ij} \tag{21}$$

where δ_{ij} is the unit matrix and λ and μ are the Lame parameters:

$$\mu = G \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{22}$$

In addition to E, ν, G, μ, and λ, the bulk modulus, K, is employed in the study of volume change under hydrostatic pressure. Let $\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$ denote the dilatation, i.e., the change in volume of an infinitesimally small cube. The pressure, p, is $\varepsilon_{kk}/3$. The bulk modulus relates the pressure to the dilatation: $p=-K \cdot \varepsilon_{kk}$, where

$$K = \frac{E}{3(1-2\nu)} \tag{23}$$