2D Continuum Problems

The document on elasticity theory for 3D continuums establishes equations for equilibrium, material law, and kinematic compatibility for a solid cube. When one dimension is removed, and the forces and deformations are within the remaining plane, we have what is called 2D continuum problems. In such problems the equilibrium and kinematic compatibility equations are trivial simplifications of the 3D equations. In contrast, the material deserves special attention.

Material Law

Hooke's law for 2D continuum problems has two versions: plane stress and plane strain.

Plane Stress

The plane stress version of the material law is the straightforward simplification of the material law for 3D problems when one coordinate is removed:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E} \\ \tau_{xy} &= G \cdot \gamma_{xy} \end{aligned} \tag{1}$$

where $G = E/\big(2\cdot(1+\nu)\big)$. This 2D material law implies that there are no stresses perpendicular to the considered plane. This applies to thin planar members that do not have stresses imposed perpendicular to the surface. One example is a deep beam that stretches along the *x*-axis with *z* as the vertical axis. Seen from the side this beam forms a 2D continuum problem in the *x*-*z*-plane with plane stress material law because there is not stress perpendicular to the sides of the beam. In Voight notation any 2D material law is written generically in index and vector notation as follows:

$$\sigma_i = D_{ij} \cdot \varepsilon_j \quad \Leftrightarrow \quad \sigma = \mathbf{D}\varepsilon \tag{2}$$

where the **D**-matrix for plane stress is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\mathbf{D}} \cdot \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{3}$$

or inversely, directly from Eq. (1):

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{4}$$

Plane Strain

Plane strain implies that there is zero *strain* perpendicular to the plane under consideration. This applies to problems where the plane is a cross-section of a structure that is long in the direction perpendicular to that plane. One example is the modelling of the cross-section of a dam structure. From the selected cross-section the dam stretches out to both sides until it meets mountainside supports. This restrains displacement, and hence strain, perpendicular to the cross-section. That plane is then in a state of plane strain, i.e., without strain perpendicular to the plane. For now without derivations this material law reads

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{5}$$

or inversely

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & 1-\nu^2 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{6}$$

Compatibility Equation in Different Forms

The 2D specialization of the 3D kinematic compatibility equations from the general theory of elasticity is trivial and reads

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{7}$$

Differentiating the first equation twice with respect to *y*, then differentiating the second equation twice with respect to *x*, and finally differentiating the third equation with respect to *x* and *y* yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \tag{8}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2} \tag{9}$$

$$\frac{\partial^2 \gamma_{xx}}{\partial y \partial x} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} \tag{10}$$

Substituting Eqs. (8) and (9) into Eq. (10) yields the "compatibility equation" that states the necessary relationship between the strains for the deformation pattern to be physically valid:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \tag{11}$$

That compatibility equation can be reformulated by the introduction of material law, here using the plane stress material law in Eq. (4), which substituted into Eq. (11) yields

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{2(1+\nu)}{E} \cdot \sigma_{xy}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\sigma_{xx}}{E} - \nu \cdot \frac{\sigma_{yy}}{E}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma_{yy}}{E} - \nu \cdot \frac{\sigma_{xx}}{E}\right) \tag{12}$$

where E cancels and the new compatibility equation simplifies to

$$\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \cdot 2 \cdot (1+\nu) = \frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu \cdot \sigma_{yy}) + \frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu \cdot \sigma_{xx}) \tag{13}$$

As shown in Section 16 of Timoshenko's book on the theory of elasticity Eq. (13) can be further modified by introducing equilibrium (Timoshenko and Goodier 1969). To achieve this the first step is to write the equilibrium equations in 2D, a straightforward simplification of the 3D version:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + f_x = 0 \tag{14}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0 \tag{15}$$

Assuming the body forces are uniform and differentiating Eq. (14) with respect to x and Eq. (15) with respect to y yields

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yx}}{\partial y \partial x} = 0 \tag{16}$$

$$\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 0 \tag{17}$$

Adding Eqs. (16) and (17), i.e., adding zero with zero yields

$$2 \cdot \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} = 0 \tag{18}$$

This equilibrium equation is now solved for , which is substituted into Eq. (13) to obtain the following new compatibility equation:

$$-\frac{1}{2} \cdot \left(\frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial x^2} \right) \cdot 2 \cdot (1+\nu) = \frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu \cdot \sigma_{yy}) + \frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu \cdot \sigma_{xx}) \tag{19}$$

Simplification yields the compatibility equation in terms of stresses

$$-\frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial^2 \sigma_{xx}}{\partial x^2} = \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} \tag{20}$$

which can be written

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\sigma_{xx} + \sigma_{yy} \right) = 0 \tag{21}$$

This equation holds also for plane strain and has a non-zero right-hand side when the body forces are not uniform (Timoshenko and Goodier 1969):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\sigma_{xx} + \sigma_{yy} \right) = -\frac{1}{1-\nu} \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) \tag{22}$$

One approach to solve 2D continuum problems is to seek stresses that imply equilibrium and that satisfy the compatibility equations presented above, together with problem-specific boundary conditions. A clever approach to achieve this is a concept called stress functions.

Stress Functions

In many boundary value problems in structural mechanics it is possible to combine equilibrium, material law, and kinematic compatibility equations into one governing differentiation equation. This equation, together with problem-specific boundary conditions on forces and displacements, are used to obtain solutions. In the 2D theory of elasticity, and certain other problems, an additional helpful concept is the “stress functions” introduced by George Biddell Airy (1801-1892) in an 1862 paper. The stress function itself does not have physical meaning. Instead it is an auxiliary quantity, here denoted $\varphi(x,y)$, from which stresses are derived:

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} - f_x \cdot x \tag{23}$$

$$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} - f_y \cdot y \tag{24}$$

$$\tau_{xy} \equiv \sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \tag{25}$$

Notice that the double derivative of the stress function in one direction produces axial stress in the perpendicular direction. That is perhaps the most physical interpretation that is possible for the stress function. Next it is interesting to combine the stress function with the equations for equilibrium, material law, and kinematic compatibility. Starting with equilibrium, Eqs. (23)-(25) are substituted into Eqs. (14) and (15) yielding

$$\frac{d\left(\frac{\partial^2 \varphi}{\partial y^2} - f_x \cdot x\right)}{dx} + \frac{d\left(-\frac{\partial^2 \varphi}{\partial x \partial y}\right)}{dy} + f_x = \frac{\partial^2 \varphi}{\partial y^2 \partial x} - f_x + -\frac{\partial^2 \varphi}{\partial x \partial y^2} + f_x = 0 \tag{26}$$

$$\frac{d\left(-\frac{\partial^2 \varphi}{\partial x \partial y}\right)}{dx} + \frac{d\left(\frac{\partial^2 \varphi}{\partial x^2} - f_y \cdot y\right)}{dy} + f_y = -\frac{\partial^2 \varphi}{\partial x^2 \partial y} + \frac{\partial^2 \varphi}{\partial x^2 \partial y} - f_y + f_y = 0 \tag{27}$$

which demonstrates that the equilibrium equations are automatically satisfied with the stress function definition in Eqs. (23)-(25). Only material law and kinematic compatibility remain and those have already been combined in the compatibility equations presented earlier. Substitution of the stress function in Eqs. Eqs. (23)-(25) into the compatibility equation in Eq. (21) and assuming uniform body forces yields

$$\frac{\partial^4 \varphi}{\partial x^4} + 2 \cdot \frac{\partial^4 \varphi}{\partial y^2 \partial x^2} + \frac{\partial^4 \varphi}{\partial y^4} = 0 \tag{28}$$

This equation is the governing equation for 2D continuum problems. Solutions are found by establishing stress functions that satisfy Eq. (28) together with problem-specific boundary conditions. Once the stress function is determined the stresses are found from Eqs. (23)-(25). However, many real in-plane members have shapes and boundary conditions that render such analytical solutions unattainable. The numerical finite element method provides and alternative in those circumstances.

References

Timoshenko, S., and Goodier, J. N. (1969). *Theory of elasticity*. McGraw-Hill.