Work and Energy Expressions

To utilize the virtual work and variational principles it is necessary to have expressions for virtual work and energy. Therefore, this document provides such expressions for a variety of situations. Consistent with the notation in other documents, *H* denotes potential energy of external load, *U* denotes potential strain energy in elastic members, *K* denotes kinetic energy, *P* denotes total potential energy, δW_{ext} denotes virtual work associated with external loads, and δW_{int} denotes virtual work associated with internal elastic deformation.

Virtual Work

Expressions for W_{int} and W_{ext} are sought for the principle of virtual displacements and the principle of virtual forces.

3D Elasticity

For the principle of virtual displacements δW_{int} of an elastic material particle is:

$$\delta W_{int} = \int_V \sigma \cdot \delta\varepsilon \, dV \tag{1}$$

For the principle of virtual forces it is:

$$\delta W_{int} = \int_V \delta\sigma \cdot \varepsilon \, dV \tag{2}$$

Beam Bending

Substitution of material law $\sigma=E\varepsilon$ and kinematics $\varepsilon=-w''z$ into Eq. (1) and integration over the cross-section, thus defining *I*, yields:

$$\delta W_{int} = \int_0^L EI \cdot w'' \cdot \delta w'' \, dx \tag{3}$$

Beam Loading

δW_{ext} for the principle of virtual displacements is force times virtual displacement:

$$\delta W_{ext} = \int_0^L q_z \, \delta w \, dx \tag{4}$$

δW_{ext} is, when inertia is included according to d'Alembert's principle:

$$\delta W_{ext} = \int_0^L \left(q_z - m \cdot \ddot{w}\right) \delta w \, dx \tag{5}$$

Load on 3D

The external virtual work associated with the forces **p** along the displacement fields $\tilde{\mathbf{u}}$ on the boundary Γ is:

$$\delta W_{ext} = \int_{\Gamma} \mathbf{p} \cdot \delta \tilde{\mathbf{u}} \, d\Gamma \tag{6}$$

Strain Energy

Strain energy in an elastic body is a form of potential energy, and potential energy quantifies the ability of a system to carry out work.

Elastic Spring

Consider a linear spring with stiffness k, force F, and elongation Δ. The force in the spring is $k\Delta$. As the spring is slowly extended, the work is the area under the load-displacement curve, identified as a shaded triangle in Figure 1, which is stored as elastic potential energy:

$$U = \int_0^{\Delta} F \, d\Delta = \int_0^{\Delta} (k \cdot \Delta) \, d\Delta = \frac{1}{2} \cdot k \cdot \Delta^2 \tag{7}$$

The complementary strain energy is expressed in terms of force instead of displacement, identified as the non-shaded triangle in Figure 1:

$$\bar{U} = \int_0^{F} \Delta \, dF = \int_0^{F} \left(\frac{F}{k} \right) dF = \frac{1}{2} \cdot \frac{1}{k} \cdot F^2 \tag{8}$$

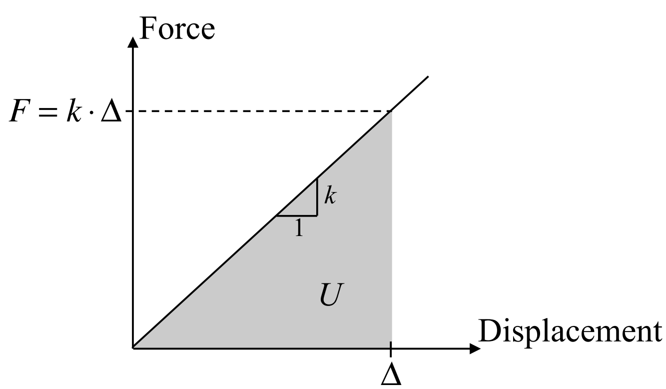

Figure 1: Strain energy.

Rotational Spring

The strain energy in a rotational spring is:

$$U = \frac{1}{2} \cdot k \cdot \theta^2 \tag{9}$$

Infinitesimal Uniaxial Stress-Strain Element

For a generic material element subjected to stress, σ, and strain, ε, the linear elastic material law is

$$\sigma = E \cdot \varepsilon \tag{10}$$

Integration with respect to strain yields:

$$U = \int_0^{\varepsilon} E \cdot \varepsilon \, d\varepsilon = \frac{1}{2} \cdot E \cdot (\varepsilon)^2 \tag{11}$$

As a result, the total strain energy for a volume of this material is

$$U = \int_V \frac{1}{2} \cdot E \cdot (\varepsilon)^2 \, dV \tag{12}$$

Bar Elongation

A straightforward application of Eq. (12) is to carry out the cross-section integration but leaving the longitudinal integration, which yields:

$$U = \int_0^L \frac{EA}{2} \cdot (\varepsilon)^2 \, dx \tag{13}$$

Substitution of kinematics yields:

$$U = \int_0^L \frac{EA}{2} \cdot (u')^2 \, dx \tag{14}$$

which could just as easily be obtained by starting with the force-deformation relationship

$$N = EA \cdot u' \tag{15}$$

which is the approach adopted next for beam bending.

Beam Bending

Analogous to the linear F-Δ relationship in Figure 1 for the derivation of the strain energy for a spring in Eq. (7), the linear moment-curve relationship

$$M = EI \cdot w'' \tag{16}$$

is considered here, although Eq. (12) could equally well be used. It is noted that kinematics, material law, and section integration is included in Eq. (16). Integration with respect to curvature yields:

$$U = \int_0^{w''} EI \cdot w'' \, dw'' = \frac{1}{2} \cdot EI \cdot (w'')^2 \tag{17}$$

As a result, the total strain energy for a beam with length L is:

$$U = \int_0^L \frac{1}{2} \cdot EI \cdot (w'')^2 \, dx \tag{18}$$

From Euler-Bernoulli beam theory it is understood that *w*'' is an approximation expression for the curvature. Eq. (17) holds valid for other curvature expressions as well, with *w*'' replaced by the alternative curvature expression. The expression for the complementary strain energy is obtained by integration along the moment axis:

$$\bar{U} = \int_0^M w'' dM = \int_0^M \frac{M}{EI} dM = \frac{1}{2} \cdot \frac{1}{EI} \cdot M^2 \tag{19}$$

Beam Bending with Large Deformations

The generic formulation of strain energy in Eq. (12) is useful when alternative strain expressions are employed. For example, for large-deformations (but still small strains) expressions for strain can be derived that include the elongation of the beam element due to transverse displacement. First consider a horizontal beam element of infinitesimal length, shown in Figure 2. While the initial position of the element is horizontal, the deformed element is shown with a thick line. The line is drawn straight because the bending deformation is addressed in the Euler-Bernoulli beam theory; here it is the elongation, denoted ΔL, which is of interest. A trigonometric consideration of the small triangle in Figure 2 yields

$$\sin\left(\frac{dw}{dx}\right) = \frac{\Delta L}{dw} \quad \Rightarrow \quad \Delta L = \sin\left(\frac{dw}{dx}\right) \cdot dw \tag{20}$$

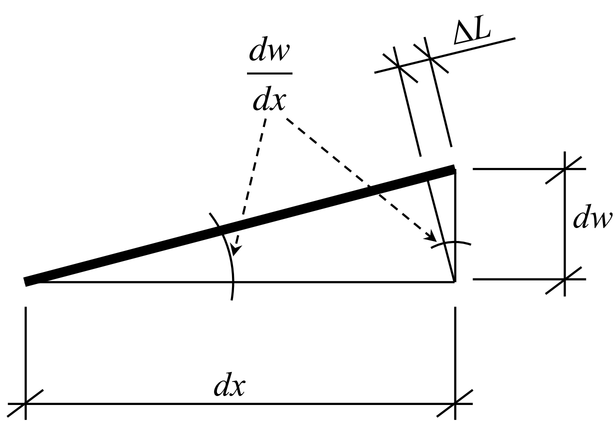

Figure 2: Elongation of beam element.

Dividing ΔL by the original length dx yields the strain:

$$\varepsilon = \sin\left(\frac{dw}{dx}\right) \cdot \frac{dw}{dx} \approx \left(\frac{dw}{dx}\right)^2 \tag{21}$$

where the approximation sin()≈() is made. In conclusion, the complete strain expression for beams that undergo large deformations, in the sense that the transverse displacement elongates the element, but still under the assumption of small strains, is:

$$\varepsilon = \frac{du}{dx} - z \cdot \frac{d^2w}{dx^2} + \left(\frac{dw}{dx}\right)^2 \tag{22}$$

This expression is referred to as Green's strain. Another version of Green's strain for a beam element is obtained for beams that have an initial camber, i.e., arches with small height. Now consider Figure 3, which identifies the original shape of the beam as *f*(*x*) so that the change in length, Δ*L*, is now obtained from a trigonometric consideration of the small triangle as

$$\sin\left(\frac{df}{dx}\right)=\frac{\Delta L}{dw} \quad \Rightarrow \quad \Delta L=\sin\left(\frac{df}{dx}\right)\cdot dw \tag{23}$$

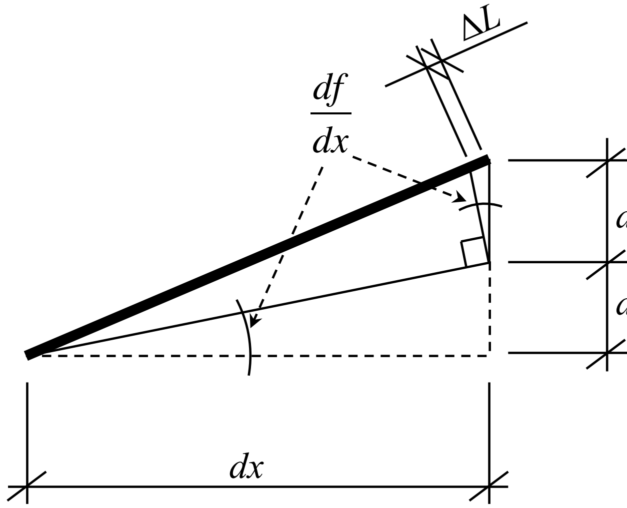

Figure 3: Elongation of beam element in a beam with initial camber.

Dividing Δ*L* by the original length *dx* yields the strain:

$$\varepsilon=\sin\left(\frac{df}{dx}\right)\cdot\frac{dw}{dx}\approx\frac{df}{dx}\cdot\frac{dw}{dx} \tag{24}$$

where the approximation sin()≈() is made because the camber is small. Thus, the complete expression for strain in this case is:

$$\varepsilon=\frac{du}{dx}-z\cdot\frac{d^2w}{dx^2}+\frac{df}{dx}\cdot\frac{dw}{dx} \tag{25}$$

which can be substituted into Eq. (12) for the analysis of such structures by the energy method. Finally, it is noted that the complete Lagrangian Green's strain for large displacement analysis of horizontal beams is as follows, often referred to as the von Karman strain from the similar expressions in plate theory:

$$\varepsilon=\frac{du}{dx}-z\cdot\frac{d^2w}{dx^2}+\frac{1}{2}\cdot\left[\left(\frac{du}{dx}-z\cdot\frac{d^2w}{dx^2}\right)^2+\left(\frac{dw}{dx}\right)^2\right] \tag{26}$$

Matrix Formulation

In matrix structural analysis the force-displacement relationship is written:

$$\mathbf{F}=\mathbf{K}\mathbf{u} \tag{27}$$

where **F** is the load vector, **K** is the stiffness matrix, and **u** is the vector of degrees of freedom. Caution must be exercised to avoid confusing the stiffness-related quantities k and **K** with the symbol K for kinetic energy in the following. The strain energy expressed in this context is:

$$U = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \tag{28}$$

Potential Energy in Loads

The archetypical expression for potential energy is $H=-P\Delta$, where the minus sign implies that potential energy is released when the load P acts along the displacement Δ. Thus, the following expressions all have a minus sign, which is removed if the force multiplied by displacement actually implies an accumulation of potential energy.

Beam Loading

The potential energy of external load on a beam is:

$$H = -\int_0^L q_z \cdot w\,dx \tag{29}$$

Matrix Formulation

The potential energy associated with the load vector, **F**, in matrix structural analysis is:

$$H = -\mathbf{F}^T\mathbf{u} \tag{30}$$

Axial Force on Rigid Column

This type of element is employed in stick models. Suppose a rigid column is subjected to a conservative axial force, P, while the element rotates by θ. By conservative it is meant that the force acts in the same direction throughout the deformation. The potential energy associated with the axial force is

$$H = -P \cdot \Delta_{vert} \tag{31}$$

where Δ_{vert} is the vertical displacement at the top of the column, as shown in Figure 4. With reference to the same figure, the vertical displacement expressed in terms of the element rotation is

$$\Delta_{vert} = L - L \cdot \cos(\theta) \tag{32}$$

In ordinary structural analysis, $\cos(\theta)$ would be considered equal to unity because θ is small. Here, however, consider the series expansion of $\cos(\theta)$:

$$\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{33}$$

The terms decrease in value as they increase in order. In the linearized second-order theory the high-order terms are neglected and

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} \tag{34}$$

As a result, the potential energy from the axial force is:

$$\begin{aligned} H &= -P \cdot \Delta_{vert} = -P \cdot \left(L - L \cdot \cos(\theta)\right) = -P \cdot \left(L - L \cdot \left(1 - \frac{\theta^2}{2}\right)\right) \\ &= -P \cdot L \cdot \frac{\theta^2}{2} \end{aligned} \tag{35}$$

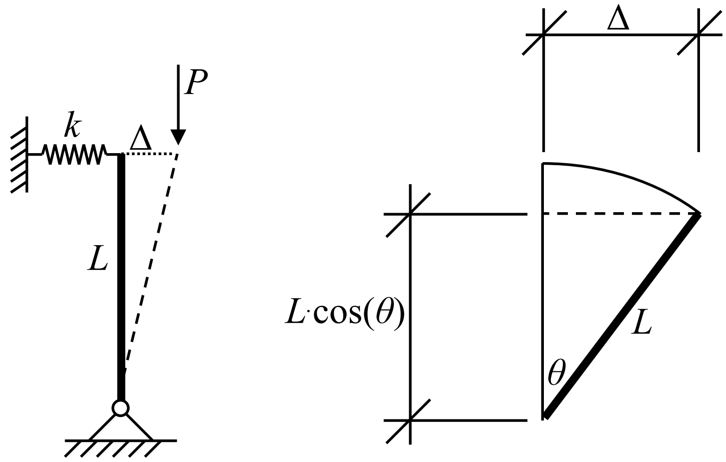

Figure 4: Rigid column subjected to axial force.

Axial Force on Beam Element

The potential energy associated with shortening du of an infinitesimally short beam element subjected to axial load is

$$H = -P \cdot du \tag{36}$$

The "shortening" du when the element displaces laterally and thus rotates by an amount θ is, similar to Figure 4:

$$du = dx - dx \cdot \cos(\theta) \tag{37}$$

Using the same approximation of cos(θ) as in Eqs. (33) and (34), and substituting $\theta = dw/dx$, yields

$$du = \frac{1}{2} \cdot \left(\frac{dw}{dx}\right)^2 dx \tag{38}$$

Consequently, the potential energy is

$$H = -\int_0^L P\, du = -P \cdot \int_0^L \frac{1}{2} \cdot (w')^2\, dx \tag{39}$$

Axial Force on Rigid Column with Geometrical Imperfection

Reconsider the rigid column in Figure 4 and the potential energy in Eq. (35). One approach to include geometrical imperfection is to include an initial deformation θ_0:

$$H = -P \cdot \Delta_{vert} = -P \cdot L \cdot \frac{(\theta - \theta_o)^2}{2} \tag{40}$$

Axial Force on Rigid Column with Load Eccentricity

To include load eccentricity for the rigid column in Figure 4 and the potential energy in Eq. (35), one approach is to include a lateral force at the top:

$$H = -P \cdot \Delta_{vert} - F \cdot \Delta = -P \cdot L \cdot \frac{\theta^2}{2} - F \cdot L \cdot \theta \tag{41}$$

Kinetic Energy

By denoting by *v* the velocity, the kinetic energy is generically written

$$K = \frac{1}{2} \cdot m \cdot v^2 \tag{42}$$

For a beam element it is written

$$K = \frac{1}{2} \cdot m \cdot \dot{w}^2 \tag{43}$$