Virtual Work and Variational Principles

Mathematically, the structural analysis problem is a boundary value problem (BVP). Forces, displacements, stresses, and strains are connected and computed within the framework of the BVP. One or more of those quantities serve as the primary unknown function, sometimes called the unknown field function. In elementary engineering education the BVP is formulated as a differential equation. It involves the formulation of equilibrium, kinematics, and material law. Combined with problem-specific boundary conditions it provides unique solutions to specific problems. However, the mechanical BVP can be formulated in other forms, which is the starting point for this document.

Forms of the BVP

Strong Form

- This form is essentially the differential equation
- This form requires point-wise satisfaction of equilibrium, kinematics, and material law

Weighted Residual Form

- This form is usually obtained by multiplying the differential equation by a weight function and integrating
- This form requires only average (integrated) satisfaction of equilibrium

Weak Form

- This form is sometimes called the virtual work form
- It is usually the principle of virtual displacements that is utilized in this context, in contrast to the principle of virtual forces, which is employed to derive the unit virtual load method
- This form is usually established either directly from the principle of virtual work or by carrying out integration by parts on the weighted residual form
- Like the weighted residual form this form requires only average satisfaction of equilibrium over the element
- In statics, this form expresses the principle of virtual work; in dynamics it is called d'Alembert's principle

Variational Form

- This form is only possible for "conservative" systems, i.e., those in which the work done by the forces is reversible; friction and follower-loads are examples of non-conservative forces, which render the variational form unattainable
- This form is sometimes called the energy form because it is based on formulations of the energy of the system

- In statics, this form expresses the principle of minimum potential energy; in dynamics it is called Hamilton's principle
- This form also requires only average satisfaction of the governing equations, but depending on the variational principle it is the equilibrium or kinematic compatibility requirements that are approximated, or both

Obviously, the strong form seems preferable because it is exact. However, it is unworkable for many practical problems. The other forms of the BVP lead to appealing "direct methods," both hand-calculation and computer methods. The accuracy of the solutions obtained by direct method depends on the selected solution functions for the unknown field.

Forms for Beam Bending

To illustrate the four forms of the mechanical BVP the problem of beam bending according to the Euler-Bernoully theory is addressed. The distributed load is denoted q_z, which is positive in the direction of the z-axis. This is different from the notation q in beam theory, which for pedagogical convenience acts downwards although the z-axis is positive upward. Consequently, the q_z in this document is the negative of the q that appears in the basic theory:

Strong form:
$$EI \cdot w'''' - q_z = 0 \tag{1}$$

Weighted residual form:
$$\int_0^L \left(EI \cdot w'''' - q_z \right) \cdot \delta w \cdot dx = 0 \tag{2}$$

Weak form:
$$\int_0^L EI \cdot w'' \cdot \delta w'' \cdot dx - \int_0^L q_z \cdot \delta w \cdot dx = 0 \tag{3}$$

Variational form:
$$\Pi(w) = \int_0^L \left(\frac{1}{2} \cdot EI \cdot \left(w'' \right)^2 - q_z \cdot w \right) dx \quad \text{with} \quad \delta\Pi = 0 \tag{4}$$

where a preceding δ identifies a virtual quantity or a variation. In particular, $\delta\Pi$ is the variation of the functional Π with respect to the function $w(x)$.

Strong Form to Variational Form, and Back

Starting at the strong form, the weighted residual form is obtained by multiplying the differential equation by a weight function, δw, which is subsequently interpreted as a virtual displacement in the weak form, and integrating over the problem domain. Next, integration by parts yields the weak form. The boundary terms that appear in the integration by part are

$$\left[EI \cdot w''' \cdot \delta w \right]_0^L - \left[EI \cdot w'' \cdot \delta w' \right]_0^L \tag{5}$$

All of these boundary terms always cancel; each term cancels because of one of two possible reasons. Either there are kinematic restrictions on the field function, w, so that δw and $\delta w'$ vanish. In other words, wherever w or w' is prescribed, e.g., as zero, there cannot be a corresponding virtual deformation. Those are called essential boundary conditions. The alternative reason that a boundary term in Eq. (5) cancels is that the remainder of the expression is zero, i.e., that EIw’’’ or EIw’’ is zero. These are called natural boundary conditions because they are part of the formulation. Notice that in this case EIw’’’ and EIw’’ represent shear force and bending moment, respectively. I.e., here the natural boundary conditions impose boundary conditions on the forces in the structure. Next, the variational form is established by equating the potential energy of the external load to the internal elastic energy. If the variational form is possible then all the other forms can be derived from it. In fact, the variation of the variational form yields the weak form, from which the weighted residual form is obtained by integration by parts. In turn, the differential equation is obtained by letting the weight function be arbitrary. When the differential equation is derived from a variational principle in this way then it is called the Euler equation.

Concepts

Vectorial and Analytical Mechanics

Starting in the late 1600’s mechanics developed in two directions, called vectorial and analytical mechanics. Both approaches are still widely used. Vectorial mechanics is prevalent in fundamental education in mechanics and in engineering practice. Analytical mechanics is more popular in advanced application of mechanics, and forms the basis for this document. A summary of the two approaches is provided in Table 1.

Table 1: Overview of vectorial and analytical mechanics.

	Vectorial mechanics	**Analytical mechanics**
Foundation	Newton’s laws	Leibniz’s vis viva (“living force”), i.e., kinetic energy, and work by Euler and Lagrange
Central quantities	Vectors: Momentum, force	Scalars: Kinetic energy and the work function (potential energy)
Coordinate system	Forces, deformation, and movement are expressed as vectors in the Euclidean geometry, i.e., a rectangular frame of reference	The scalars are invariant to the frame of reference
Formulation	Equations	Principles

Principles and Direct Methods

As shown in Table 1, the concept of energy is central in analytical mechanics. This explains the title of this document (“Energy Methods”). Table 1 also shows that analytical mechanics is formulated in terms of principles. The principles establish fundamental physical insight while the corresponding methods are devised to calculate solutions for specific problems. An overview of some of the principles and their associated solution methods, usually called direct methods, is provided in Table 2. The table also identifies the form of the BVP for each principle.

Table 2: Overview of principles and associated methods.

Principle	Significance	BVP form	Direct Methods
Principle of virtual work		Weak form	• Rayleigh-Ritz method • Finite element method
d'Alembert's principle	Adds dynamics to the principle of virtual work	Weak form	•
Principle of minimum potential energy	Implies equilibrium; deformations are unknown	Variational form	• Rayleigh-Ritz method • Finite element method
Hamilton's principle	Adds dynamics to the principle of minimum potential energy	Variational form	•
Principle of complementary virtual work		Variational form	•
Principle of minimum complementary energy	Implies kinematic compatibility; forces are unknown	Variational form	•
Hu-Washizu's principle	Includes both equilibrium and kinematics; both forces and deformations are unknown	Variational form	•
Hellinger-Reissner's principle	Includes both equilibrium and kinematics; both forces and deformations are unknown	Variational form	•
N/A		Weighted residual form	• Galerkin method • Least squares method • Finite element method
N/A		Strong form	• Analytical solutions • Finite differences

Conservative and Non-conservative Systems

A mechanical system is called conservative if expended work is recoverable. Examples of non-conservative systems are problems with friction force or follower-loads, i.e., loads that change direction during deformation. An important significance of conservative systems is that the variational form of the BVP exists.

Calculus of Variations

While ordinary calculus deals with functions of one or more variables the calculus of variations deals with functions of functions. Often, the quantity at hand is an integral of a function, in which the function that minimizes the integral is sought. Leonard Euler employed this type of formulation, specifically the potential energy for beam bending established by his contemporary Daniel Bernoulli, to solve beam-bending problems. Such functions of functions are called functionals. The calculus of variations, which is outlined later in this document, establishes rules that have similarities to ordinary calculus.

Types of Boundary Conditions

In vectorial mechanics, boundary conditions are specified by restricting deformations and/or forces at particular locations in the structure. This is of course necessary in analytical mechanics as well, but here the boundary conditions are categorized in a mathematical framework. Natural boundary conditions are named so because they are part of the principle, while essential boundary conditions must be prescribed as part of the solution. Which boundary conditions are contained in the principle depends on the principle. As an example, the principle of minimum potential energy contains the mechanical, i.e., force, boundary conditions as the natural boundary conditions. Conversely, Hellinger-Reissner's principle contains both kinematic and force boundary conditions as natural boundary conditions. This terminology should not be confused with the one from solution of differential equations, in which Dirichlet boundary conditions specify the value of a function on a surface and Neumann boundary conditions specify the normal derivative of a function on a surface.

Generalized Coordinates

The concept of generalized coordinates, **q**, is similar to the use of degrees of freedom, **u**, in the stiffness method. However, instead of representing actual nodal displacements or rotations each generalized coordinate is a multiplier for a function. Each function is usually a shape function that mimics the deformation of the structure. The concept can also be used to model force fields. Generalized coordinates are applied in the development of some of the finite elements in these notes. In this document, they are utilized in the direct methods. To this end, the ordinary load vector, **F**, and displacement vector, **u**, are have the generalized counterparts **Q** and **q**. Consider, as an example, the discretization of the beam deflection function, $w(x)$:

$$w(x) = \mathbf{N}_q(x)\mathbf{q} \tag{6}$$

where $\mathbf{N}_q$ contains the shape functions that correspond to each generalized coordinate **q**. Caution must be exercised to avoid confusion with the notation q, q_x, q_y, and q_z, for distributed loads on beams. For now it is noted that the generalized shape functions must represent a complete set of orthogonal functions.

Principles

Principle of Virtual Displacements

In fundamental structural analysis the principle of virtual forces is utilized to derive the unit virtual load method, as described in another document. However, the principle of virtual displacements plays an even greater role in advanced structural mechanics. It states that a deformable body is in equilibrium if the external virtual work equals the internal virtual work for any virtual deformation that satisfies the kinematic boundary conditions:

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \tag{7}$$

where the expressions for dWint and dWext contain virtual deformations, such as strain, displacement, and curvature. Expressions for virtual work are provided in another document. Those expressions do NOT have a ½-factor because the virtual displacements are applied first, and carry out work when the actual displacements are imposed.

d'Alembert's Principle

This principle is the principle of virtual work extended with an inertia term. d'Alembert states that according to Newton's second law there will be an inertia force in the opposite direction compared to the direction of the acceleration. Figure 1 illustrates this for a generic single-degree-of-freedom system on the left and an infinitesimal beam element on the right. Consequently, the extension of Eq. (7) for beam problems according to d'Alembert's principle reads:

$$\begin{gathered}\delta W_{\text{int}} = \delta W_{\text{ext}} \\ \Downarrow \\ \int_V \sigma \cdot \delta\varepsilon \, dV = \int_\Gamma \left(\mathbf{p} - m \cdot \ddot{\tilde{\mathbf{u}}} \right) \cdot \delta\tilde{\mathbf{u}} \, d\Gamma\end{gathered} \tag{8}$$

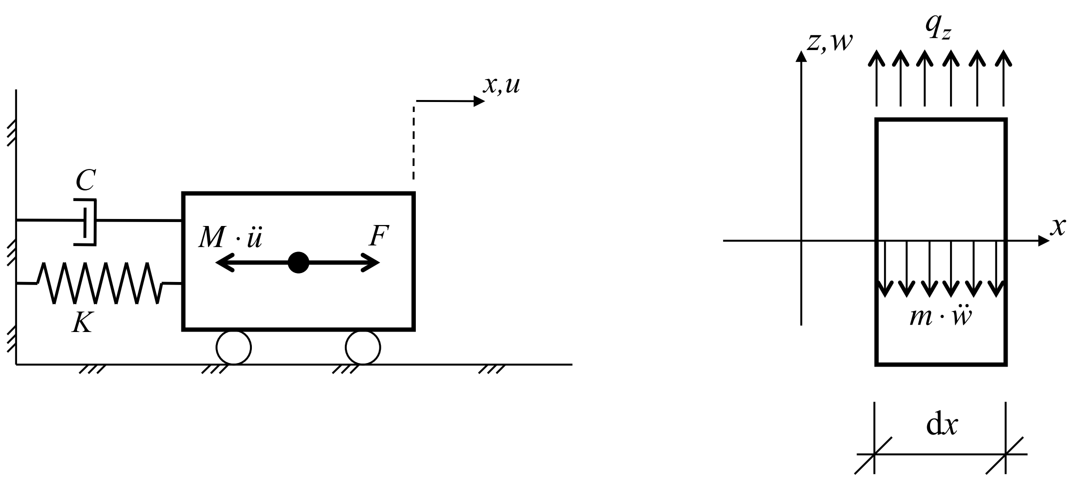

Figure 1: d'Alembert's principle.

Principle of Minimum Potential Energy

This principle can be derived from the principle of virtual displacements. Thus, it represents integrated equilibrium. It can also be derived from the laws of thermodynamics. The principle states that of all displacements that satisfy kinematic boundary conditions the ones that minimize the potential energy provide stable static equilibrium. The principle is applied by expressing the potential energy associated, *H*, associated with external loads, and the elastic strain energy, *U*, and adding them to obtain the total potential energy:

$$\Pi = H + U \tag{9}$$

Equilibrium requires a stationary value of the functional, i.e., that the first variation of the functional is zero:

$$\delta\Pi = 0 \tag{10}$$

and stable equilibrium requires that the second-variation is greater than zero for all variables

$$\delta^2 \Pi > 0 \tag{11}$$

i.e., the equilibrium is stable when the potential energy increase in the vicinity of the solution.

Castigliano's Theorems

Castigliano's first theorem states that if the strain energy, U, can be expressed in terms of a set of generalized displacements, $\mathbf{q}$, that are associated with the generalized forces, $\mathbf{Q}$, then any force can be computed by partial integration of the strain energy with respect to the corresponding displacement:

$$Q_i = \frac{\partial U(\mathbf{q})}{dq_i} \tag{12}$$

Conversely, Castigliano's second theorem states that

$$q_i = \frac{\partial U(\mathbf{Q})}{dQ_i} \tag{13}$$

Hamilton's Principle

This principle is the same as the principle of minimum potential energy, but extended with an inertia term. In other words, Hamilton's principle is in dynamics what the principle of minimum potential energy is in statics. Hamilton's principle can be derived from d'Alembert's principle. It can also be established as an expression for the conservation of mechanical energy, which states that the work equals the change in energy. To this end, consider the following derivation of the work-energy theorem. The incremental work carried out by a force F as it moves a particle by an infinitesimal displacement du is:

$$dW = F \cdot du \tag{14}$$

Next, introduce the rate of change of the displacement over an infinitesimal time period dt:

$$dW = F \cdot \frac{du}{dt} \cdot dt \equiv F \cdot \dot{u} \cdot dt \tag{15}$$

Furthermore, apply Newton's second law to express the force, F, in terms of the particle's acceleration:

$$F = m \cdot \ddot{u} \tag{16}$$

which yields the work-energy theorem:

$$dW = F \cdot \dot{u} \cdot dt = m \cdot \ddot{u} \cdot \dot{u} \cdot dt = \frac{d}{dt}\left(\frac{1}{2} \cdot m \cdot \left(\dot{u} \right)^2 \right) \cdot dt \tag{17}$$

Eq. (17) expresses that the work over an infinitesimal displacement equals the rate of change of the kinetic energy. By integration over time it is understood that the work equals the integral of all changes in the kinetic energy, i.e., the total change in the kinetic energy between two time instants. More generally, the conservation of mechanical energy holds:

$$\Delta K + \Delta \Pi = W \tag{18}$$

where Δ denotes change between two time instants. Hamilton's principle formulates this for conservative systems, i.e., in which all work is recoverable and included in K and Π by the action functional:

$$A = \int_{t_1}^{t_2} (K - \Pi) dt \tag{19}$$

where the integrand K–Π is called the Lagrange function. It is required that for dynamic equilibrium the variation must be zero:

$$\delta A = 0 \tag{20}$$

Essentially, Hamilton's principle requires that, for conservative systems, the sum of kinetic and potential energy remains constant. For non-conservative systems Hamilton's principle is extended to include the work:

$$\delta A = \int_{t_1}^{t_2} (\delta K + \delta C + \delta W) dt \tag{21}$$

where C represent Lagrange multipliers on the present constraints. The potential energy does not appear because the forces cannot be derived from it for non-conservative problems. In circumstances where some forces are conservative and others are not then the conservative ones can be derived from the potential energy and the non-conservative ones from the virtual work, δW, that they are associated with.

Principle of Virtual Forces: Complementary Virtual Work

The principle of virtual forces, sometimes called the principle of complementary virtual work, states that the external and internal work carried out by virtual forces that are placed on the structure before the actual load is applied are equal:

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \tag{22}$$

where the expressions for dWint and dWext contain virtual forces, such as loads, stresses, and moments. Expressions for virtual work are provided in another document. Those expressions do NOT have a ½-factor because the virtual load(s) are applied first, and carry out work when the actual displacements are imposed.

Principle of Minimum Complementary Energy

In this principle the energy is expressed in terms of forces instead of displacements. A bar denotes the complementary quantities and the principle reads

$$\bar{\Pi} = \bar{H} + \bar{U} \tag{23}$$

and equilibrium still requires that:

$$\delta\bar{\Pi} = \delta\bar{H} + \delta\bar{U} = 0 \tag{24}$$

Extended Principle of Minimum Potential Energy

Consider a modification of the principle of minimum potential energy, in which the kinematic boundary conditions are included by application of Lagrange multipliers.

Hu-Washizu's Principle

Further extend the principle of minimum potential energy by including both kinematic compatibility and kinematic boundary conditions by application of Lagrange multipliers.

Hellinger-Reissner's Principle

Modify Hu-Washizu's complete principle by converting it to complementary strain energy, resulting in a function with both deformations and stresses and unknown field functions. In this formulation, separate interpolation of deformations and stresses is possible. In other words, this principle does not prioritize either kinematic compatibility or equilibrium.

Direct Solution Methods

The finite element method is the most popular application of the virtual work and variational principles. It is described in other documents. Here, other direct methods that are workable even in hand calculations are described.

Selection of Solution Functions

The key notion in direct methods is that the unknown field function is approximated by a set of complete and linearly independent solution functions as conceptualized in Eq. (6).

Galerkin Method

The Galerkin method employs the weighted residual form of the BVP. Because this form can be established for all BVPs it is the most versatile approach. However, this method is also the strictest in terms of selection of solution functions because they must satisfy both natural and essential boundary conditions.

Rayleigh-Ritz Method

The Rayleigh-Ritz method employs the variational form or the virtual work form of the BVP. Because those forms are not always available it is a more restricted method. However, the selection of potential solution functions is more flexible. While the weight function in the Galerkin approach must satisfy all the same boundary conditions as the actual displacement function, the Rayleigh-Ritz functions need only satisfy the essential boundary conditions. For the principle of virtual displacements and the principle of minimum potential energy these are the kinematic boundary conditions, i.e., those on the displacements and rotations. For the same shape functions the Galerkin method and the Rayleigh-Ritz method produce the same results.