Unit Virtual Load Method

This is a powerful method for calculating displacements and rotations, primarily in statically determinate structures. Compared with the moment-area method, the analysis procedure in the virtual unit load method is more generic. The same procedure is applied to all structures.

In textbooks, this method is often referred to simply as the virtual work method. However, the principle of virtual work is more broadly applicable than in the application of a unit load to compute deformations. Therefore, a more specific title is adopted here.

The word "virtual" may prompt the question whether there is a "real work method." There is. It is sometimes referred to as the work-energy method. However, the method of real work has serious limitations; it only provides the deformation for structures loaded with one point load, exactly at the location where that load is acting. Conversely, the virtual work method let us determine the displacement and rotation at any location for any load pattern.

The unit virtual load method is most convenient when applied to statically determinate structures. The reason is that the method requires a re-analysis of the structure to determine the bending moment diagram for a unit force or moment applied to the structure at the location where the displacement or rotation is sought. This extra analysis is of course possible for any structure, e.g., by computer methods, but it is most convenient for statically determinate structures.

The derivation of the unit virtual load method starts by studying the concept of work. Work is defined as force multiplied with displacement, or equivalently, moment multiplied with rotation. When the force is constant during the displacement then the work is:

$$W = F \cdot \Delta \quad \text{or} \quad W = M \cdot \theta \tag{1}$$

where W is work, F is force, Δ is displacement, M is moment, and θ is rotation. If the force varies during the deformation then the work is evaluated by integration:

$$W = \int_0^{\Delta} F \;\mathrm{d}\Delta \quad \text{or} \quad W = \int_0^{\kappa} M \;\mathrm{d}\theta \tag{2}$$

These integrals are illustrated in Figure 1 for nonlinear material behaviour (left) and linear material behaviour (right). For the latter case, i.e., Hooke's Law, the resulting work always contains the factor ½:

$$W = \frac{1}{2} \cdot F \cdot \Delta \quad \text{or} \quad W = \frac{1}{2} \cdot M \cdot \theta \tag{3}$$

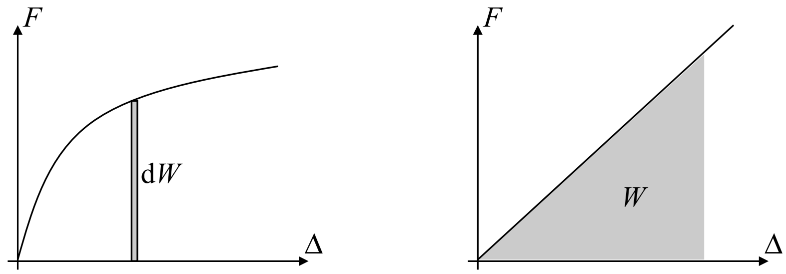

Figure 1: Accumulation of work.

Turning to internal work, U, in a structural member, the principle is the same. However, now the work is integrated over the entire member. In general, this integral reads

$$U = \int_V \frac{1}{2} \cdot \sigma \cdot \varepsilon \cdot \mathrm{dV} \tag{4}$$

where V is the volume of the structural member. First, consider a truss member with length, L, constant cross-section area, A, modulus of elasticity, E, and a constant axial force, N. By introducing the fundamental equations from truss member theory, the internal work reads:

$$\begin{aligned} U &= \int_V \frac{1}{2} \cdot \sigma \cdot \varepsilon \cdot \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{N}{A}\right) \cdot \left(\frac{\sigma}{E}\right) \cdot \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{N}{A}\right) \cdot \left(\frac{N}{EA}\right) \cdot \mathrm{d}V \\ &= \int_0^L \frac{1}{2} \cdot N \cdot \left(\frac{N}{EA}\right) \cdot \mathrm{d}x = \frac{1}{2} \cdot \underbrace{N}_{\text{Force}} \cdot \underbrace{\left(\frac{N}{EA}\right) \cdot L}_{\text{Elongation}} \end{aligned} \tag{5}$$

where the material law, $\sigma = E\varepsilon$, and the section force resultant, $N = \sigma A$, have been introduced. In contrast, it is unnecessary to introduce the kinematic equation for the member when an expression is sought in terms of the internal forces of the member. Similarly, for a frame member subjected to bending, the internal work is:

$$\begin{aligned} U &= \int_V \frac{1}{2} \cdot \sigma \cdot \varepsilon \, \mathrm{d}V = \int_V \frac{1}{2} \cdot \left(\frac{M}{I} \cdot z\right) \cdot \left(\frac{\sigma}{E}\right) \mathrm{d}V \\ &= \int_V \frac{1}{2} \cdot \left(\frac{M}{I} \cdot z\right) \cdot \left(\frac{M}{EI} \cdot z\right) \mathrm{d}V = \int_0^L \frac{1}{2} \cdot \underbrace{M}_{\text{Moment}} \cdot \underbrace{\left(\frac{M}{EI}\right)}_{\text{Curvature}} dx \end{aligned} \tag{6}$$

In addition to the internal work from axial force and bending moment it is also possible to consider internal work due to shear force. This facilitates the inclusion of shear deformation in advanced structural analysis. In accordance with the classical theory for inclusion of shear deformations (see the document on Timoshenko Beam Theory) an "average shear strain," γ_v, is introduced. It is called average because the shear strain varies over the cross-section, proportional to the shear stresses. From elementary beam member theory we know that the shear stresses are not constant over the cross-section, hence neither are the shear strains. To simplify matters, the average shear strain (sometimes called shear angle) is introduced. In the beam theory the average shear strain is related to the "shear area" A_v of the cross-section. Consequently, the internal work due to shear force is

$$U = \int_0^L V \cdot \gamma_v \, \mathrm{d}x = \int_0^L V \cdot \frac{\tau_v}{G} \, \mathrm{d}x = \int_0^L V \cdot \frac{V}{G \cdot A_v} \, \mathrm{d}x \tag{7}$$

Having established expressions for external work due to applied forces/moments, as well as internal work in structural members, attention turns to the concept of virtual work. The "secret" of virtual work is understood by considering one of its principles: the principle of virtual forces. Imagine a structure loaded by its actual loads as well as some virtual (imaginary, dummy) loads. In fact, imagine that the virtual loads are placed on the structure first. Thereafter the actual loads are gently placed on the structure. The situation is illustrated in Figure 2, in terms of external and internal work. In the figure and throughout these notes, virtual forces and deformations are identified by a preceding delta.

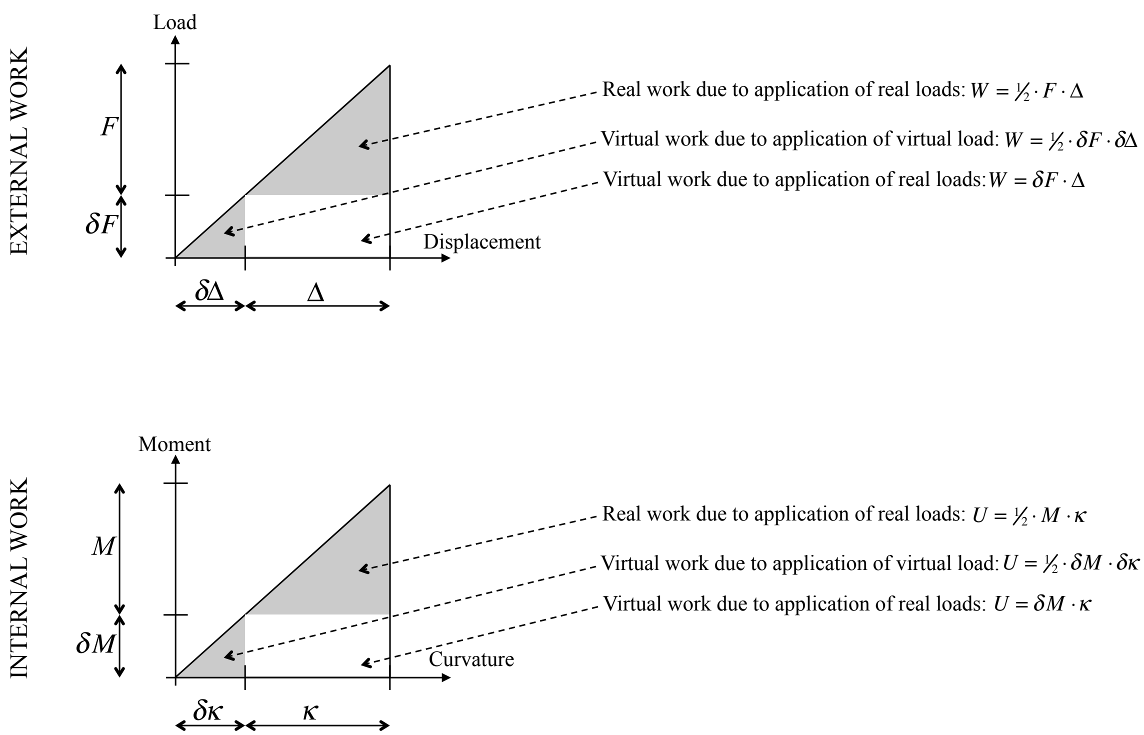

Figure 2: The principle of virtual work.

Figure 2 identifies by shaded areas the virtual work that is carried out when the virtual loads are applied, and the real work that is carried out when the real loads are applied. More importantly, it shows by white rectangles the virtual work that is carried out because the virtual forces are present when the real loads are applied. Because of the rectangular shape, this work does not contain the ½ factor.

The principle of conservation of energy states that the shaded triangles due to the application of the virtual load, i.e., the external and internal work, must be equal. It also states that the shaded triangles due to the application of the real loads, i.e., the external and internal work, must be equal. Consequently, the rectangles in Figure 2 must also be identical. This leads to the principle of virtual forces:

$$\left.\begin{matrix} \delta F \cdot \Delta \\ \delta M \cdot \theta \end{matrix}\right\} = \sum_{\substack{\text{Sum over} \\ \text{all members}}} \left(\frac{\delta N \cdot N \cdot L}{EA} + \int_0^L \frac{\delta M \cdot M}{EI} \mathrm{d}x + \int_0^L \frac{\delta V \cdot V}{G \cdot A_v} \, \mathrm{d}x \right) \tag{8}$$

In practical evaluation of Eq. (8), shear deformations, i.e., the last term, is usually omitted unless the cross-section is high compared to the length of the beam. Regardless, the procedure to determine deformations by the unit virtual load method is:

1. Determine the section force diagrams (M, N, V) due to the real loads.
2. Determine the section force diagrams due to a unit virtual force/moment placed where the displacement/rotation is sought. It is set equal to 1.0 for convenience.
3. Evaluate the right-hand side of Eq. (8) by summing contributions from all members of the structure. Quick integration formulas are provided in an auxiliary document.
4. Because the virtual force is set to unity, the result is the sought displacement or rotation.

Settlements and Changes in Member Lengths

The virtual work approach can also be utilized to determine the displacement and rotation at any location in the structure due to settlements and change in the length of members due to, e.g., temperature change and fabrication errors. In this context there is a significant difference between statically determinate and indeterminate structures. Statically determinate structures do NOT develop internal forces under these circumstances. Conversely, settlements and member length changes do generally develop internal forces due to such effects. This is a key difference between determinate and indeterminate structures.

Support settlements are first addressed. In accordance with the earlier derivations, the settlements occur after the virtual load is applied. As a result, the total external virtual work is

$$\delta F \cdot \Delta + \delta F_{S1} \cdot \Delta_{S1} + \delta F_{S2} \cdot \Delta_{S2} + \cdots \tag{9}$$

where the first term is the always-present work due to the unit load, while the other terms are due to settlements Δ_{Si} at supports where the reaction forces due to the unit virtual load is δF_{Si}. The expression in Eq. (9) replaces the left-hand side in Eq. (8). Conversely, the effect of change in member length affects the right-hand side of Eq. (8). In the original right-hand side, the internal work due to axial deformation is

$$\delta N \cdot \left(\frac{N \cdot L}{EA} \right) \tag{10}$$

where the parenthesis represents the axial deformation. The effect of temperature changes and fabrication errors are readily included by amending that expression:

$$\delta N \cdot \left(\frac{N \cdot L}{EA} + \alpha \cdot \Delta T \cdot L + \Delta L_{\text{fab. error}} \right) \tag{11}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change. $\alpha \approx 1.2 \cdot 10^{-5} °C^{-1}$ for steel and concrete. If the temperature varies over the cross-section then it causes, in general, both curvature and an overall change in the length of the member. Let ΔT_{top} and ΔT_{bottom} denote the temperature change on each side. Then, the change in member length is obtained by averaging the temperature on each side:

$$\Delta L = \alpha \cdot \left(\frac{\Delta T_{top} + \Delta T_{bottom}}{2} \right) \cdot L \tag{12}$$

The curvature from differential temperature change is computed by first recalling the relationship between strain and curvature when the strain is symmetric about the neutral axis:

$$\varepsilon = \kappa \cdot \frac{h}{2} \tag{13}$$

where h is the height of the cross-section. In this case, the strain in the outer fibre is what is left over after the average temperature change in Eq. (12) is subtracted:

$$\begin{aligned} \varepsilon &= \alpha \cdot \left(\Delta T_{top} - \Delta T_{average} \right) = \alpha \cdot \left(\Delta T_{top} - \left(\frac{\Delta T_{top} + \Delta T_{bottom}}{2} \right) \right) \\ &= \alpha \cdot \left(\frac{\Delta T_{top} - \Delta T_{bottom}}{2} \right) \end{aligned} \tag{14}$$

Combination of Eqs. (13) and (14) yields the curvature from differential temperature change, which is added to the ordinary bending curvature from the load:

$$\int_0^L \delta M \cdot \left(\frac{M}{EI} + \alpha \cdot \frac{\left(\Delta T_{top} - \Delta T_{bottom} \right)}{h} \right) \cdot \mathrm{d}x \tag{15}$$

In practical applications of the unit virtual load method, the following sign convention applies: If the curvature (or change in length) is in the same direction as the curvature (or change in length) of the virtual moment (or axial load) then the contribution is positive. The directions are opposite then the contribution is negative. This is emphasized by a ±-sign in the following summarizing amended version of Eq. (8):

$$\begin{pmatrix} \delta F \cdot \Delta \\ +\delta F_{S1} \cdot \Delta_{S1} \\ +\delta F_{S2} \cdot \Delta_{S2} \\ +\cdots \end{pmatrix} = \sum_{\substack{\text{Sum over} \\ \text{all members}}} \begin{pmatrix} \delta N \cdot \left(\frac{N \cdot L}{EA} + \alpha \cdot \Delta T \cdot L + \Delta L_{\text{fab. error}} \right) \\ + \int_0^L \delta M \cdot \left(\frac{M}{EI} \pm \alpha \cdot \frac{\left| \Delta T_{top} - \Delta T_{bottom} \right|}{h} \right) dx \\ + \int_0^L \frac{\delta V \cdot V}{G \cdot A_v} \, \mathrm{d}x \end{pmatrix} \tag{16}$$

This equation is also valid if the left-hand side consists of moments rather than forces. When inserting changes in member lengths, i.e., $\Delta L_{\text{fab.error}}$ recall that, in general, compression is negative and tension is positive, which implies that shortening is negative and lengthening is positive.