Timoshenko Beams

The Euler-Bernoulli beam theory neglects shear deformations by assuming that plane sections remain plane and perpendicular to the neutral axis during bending. As a result, shear strains and stresses are removed from the theory. Shear forces are only recovered later by equilibrium: $V=dM/dx$. In reality, the beam cross-section deforms somewhat like what is shown in Figure 1c. This is particularly the case for deep beams, i.e., those with relatively high cross-sections compared with the beam length, when they are subjected to significant shear forces. Usually the shear stresses are highest around the neutral axis, which is where; consequently, the largest shear deformation takes place. Hence, the actual cross-section curves. Instead of modelling this curved shape of the cross-section, the Timoshenko beam theory retains the assumption that the cross-section remains plane during bending. However, the assumption that it must remain perpendicular to the neutral axis is relaxed. In other words, the Timoshenko beam theory is based on the shear deformation mode in Figure 1d.

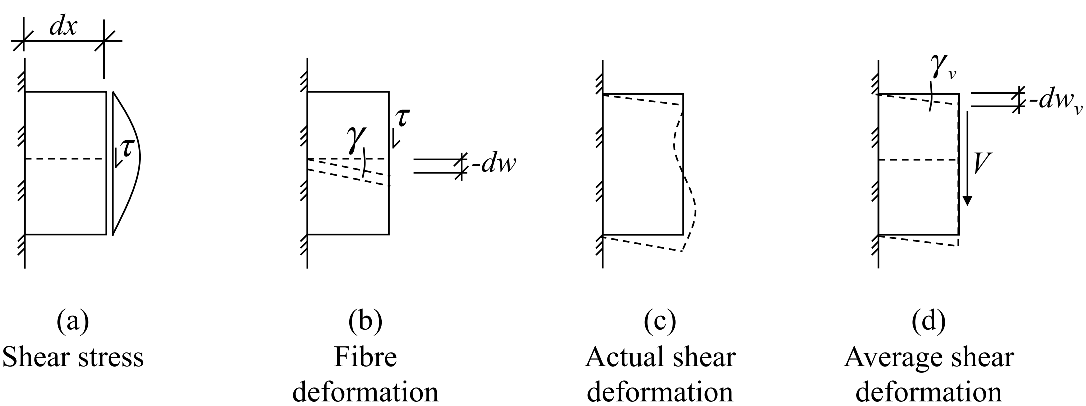


Figure 1: Shear deformation.

The average shear deformation in Figure 1d is linked with reality, i.e., the shear deformation in Figure 1b and Figure 1c by equality of work. It is required that the work carried out in the average deformation must equal the sum of the work carried out by all fibres deforming:

$$\int_A \frac{1}{2} \cdot dw \cdot \tau \; dA = \frac{1}{2} \cdot dw_v \cdot V \tag{1}$$

Substitution of the kinematic relationship $w=\gamma dx$ yields:

$$\int_A \frac{1}{2} \cdot (\gamma \cdot dx) \cdot \tau \; dA = \frac{1}{2} \cdot (\gamma_v \cdot dx) \cdot V \tag{2}$$

Substitution of the material law $\tau=G\gamma$, where G is the shear modulus defined by $G=E/(2(1+\nu))$, yields:

$$\int_A \frac{1}{2}\cdot\left(\frac{\tau}{G}\right)\cdot dx\cdot\tau\ dA = \frac{1}{2}\cdot\left(\frac{\tau_v}{G}\right)\cdot dx\cdot V \tag{3}$$

Furthermore, on the right-hand side the average shear stress is written in terms of the shear force on the cross-section, i.e., $\tau_v=V/A_v$, where A_v is an auxiliary shear area that is defined shortly:

$$\int_A \frac{1}{2}\cdot\left(\frac{\tau}{G}\right)\cdot dx\cdot\tau\ dA = \frac{1}{2}\cdot\left(\frac{V}{A_v}\right)\cdot\frac{1}{G}\cdot dx\cdot V \tag{4}$$

Substitution of the expression for shear stress from Euler-Bernoulli beam theory on the left-hand side, and definition of the shear area as $A_v=\beta A$, where β is a constant that is defined shortly yields:

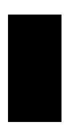
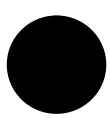
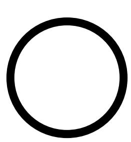
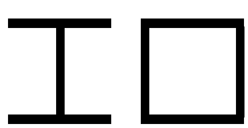
$$\int_A \frac{1}{2}\cdot\frac{1}{G}\cdot dx\cdot\left(\frac{V\cdot Q}{I\cdot t}\right)^2 dA = \frac{1}{2}\cdot\frac{V}{\beta\cdot A}\cdot\frac{1}{G}\cdot dx\cdot V \tag{5}$$

Solving Eq. (5) for β yields:

$$\beta = \frac{I^2}{A\cdot\int_A\left(\frac{Q}{t}\right)^2 dA} \tag{6}$$

Essentially, β reveals how much of the cross-section that the shear force is “smeared” uniformly over in the average shear deformation configuration. The value of β for a few typical cross-sections is provided in Table 1.

Table 1: Value of β for some cross-section shapes.

Shape	β
	5/6
	9/10
	1/2
	$\sim A_{web}/A$

Upon determining the shear area, A_v, the shear deformation is readily included in the structural analysis, as shown in the unit virtual load method for deformation computations. Notice in particular that, according to the equations written in the text above, the shear angle contribution from each infinitesimally long beam element is

$$\gamma_v = \frac{\tau_v}{G} = \frac{V}{GA_v} \tag{7}$$

where $A_v=\beta A$. Eq. (7) is employed in the unit virtual load method to include shear deformation in the calculation of displacements and rotations. That is the straightforward approach to determine the shear deformation for various beams cases.

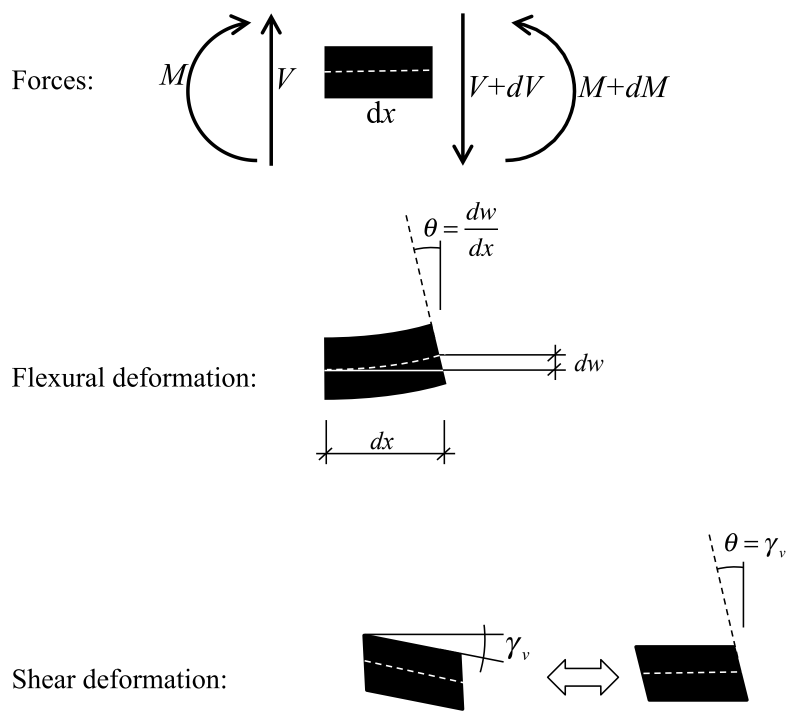


Figure 2: Contribution from shear deformation to cross-section rotation.