The Truss Element

The finite element method is here illustrated for the truss element in Figure 1. As shown, the element in its local configuration has two DOFs. Transformation to a global structural coordinate system is addressed in the document on the computational stiffness method. For this element the stiffness matrix is known from the classical stiffness method. For reference, it is

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \tag{1}$$

Figure 1: Truss element.

In the following, it is shown that the finite element method produces an integral expression for the stiffness matrix. Several starting points are possible, including variational principles. In this document it is arbitrarily selected to employ the weak form of the structural mechanics boundary value problem, i.e., the principle of virtual work. Moreover, it is decided to derive the principle by starting with the strong form, i.e., the differential equation. To this end, consider the differential equation for a truss element, i.e., the strong form of the BVP:

$$EA\tilde{u}'' + q_x = 0 \tag{2}$$

Multiplication by a weight function, $\delta\tilde{u}$, and integration over the length, L, of the element yields

$$\int_0^L \left(EA\tilde{u}'' + q_x\right)\delta\tilde{u}\,dx = \int_0^L EA\tilde{u}''\delta\tilde{u}\,dx + \int_0^L q_x\,\delta\tilde{u}\,dx = 0 \tag{3}$$

Integration by parts is applied to the term with uneven order of differentiation:

$$\int_0^L EA\tilde{u}''\delta\tilde{u}\,dx = \left[EA\tilde{u}'\delta\tilde{u}\right]_0^L - \int_0^L EA\tilde{u}'\delta\tilde{u}'\,dx \tag{4}$$

The boundary term contains the weight function, which can be interpreted as a virtual displacement field, and the derivative of the displacement, which is proportional to the axial force in the member. That is, the boundary term contains real force multiplied by virtual displacement. Because the virtual displacement must obey kinematic boundary conditions, either the force or the virtual displacement is zero at the element ends. Consequently, the boundary term cancels. The following remains:

$$-\int_0^L EA\tilde{u}'\delta\tilde{u}'\,dx + \int_0^L q_x\,\delta\tilde{u}\,dx = 0 \tag{5}$$

This is the weak form of the BVP for truss elements. The crucial step in the derivation of the finite element method follows, i.e., the substitution of the shape function discretization. Imagine the truss element in a horizontal position and suppose has two DOFs, the first, u_1, at the left end where x=0 and the other, u_2, at the right end where x=L. Then the shape function discretization reads

$$\tilde{u}(x) = \mathbf{Nu} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6}$$

where

$$\begin{aligned} N_1(x) &= 1 - \frac{x}{L} \\ N_2(x) &= \frac{x}{L} \end{aligned} \tag{7}$$

Notice the general characteristic of all shape functions: they are equal to unity at the node that contains the DOF that the shape function is associated with, and zero at the other nodes. Substitution of Eq. (6) into Eq. (5) and employing the same discretization of the real and the virtual displacement field yields

$$-\int_0^L EA(\mathbf{N}'\mathbf{u})(\mathbf{N}'\delta\mathbf{u})\,dx + \int_0^L q_x(\mathbf{N}\delta\mathbf{u})\,dx = 0 \tag{8}$$

Because the transpose of a scalar is the same scalar, rearranging yields

$$\delta\mathbf{u}^T\left(-\left(\int_0^L EA\cdot\mathbf{N}'^T\,\mathbf{N}'\,dx\right)\mathbf{u} + \int_0^L q_x\mathbf{N}^T\,dx\right) = 0 \tag{9}$$

Because the virtual displacement field, i.e., the weight function, is arbitrary the content of the large parenthesis must be zero, and consequently

$$\underbrace{\left(\int_0^L EA\cdot\mathbf{N}'^T\,\mathbf{N}'\,dx\right)}_{\mathbf{K}}\mathbf{u} = \underbrace{\int_0^L q_x\mathbf{N}^T\,dx}_{\mathbf{F}} \tag{10}$$

where the stiffness matrix and the load vector are identified. Substitution of Eq. (7) into Eq. (10) and assuming that q_x is constant in this demonstration yields

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{q_xL}{2} \\ \frac{q_xL}{2} \end{Bmatrix} \tag{11}$$

This result is equal to that obtained in the classical stiffness method. The reason is that the shape functions that were adopted match the solution to the differential equation for a truss element. When that is possible then the finite element method provides exact results.

Geometric Stiffness

Another document on geometric nonlinearity describes the concept of geometric stiffness. Based on the principles outlined there, the geometric stiffness matrix for truss elements is straightforward. Truss elements remain straight during deformation of the structure, hence the P/L term that is observed as geometric stiffness for a stick model is sufficient:

$$\mathbf{k} = \frac{EA}{L} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \underbrace{\frac{P}{L} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\mathbf{k}_G} \tag{12}$$

Note Eq. (12) is established for a truss element with four degrees of freedom, as shown in Figure 2. This is necessary because P-delta effects can only be introduced for elements with DOFs that can describe rigid-body motions.

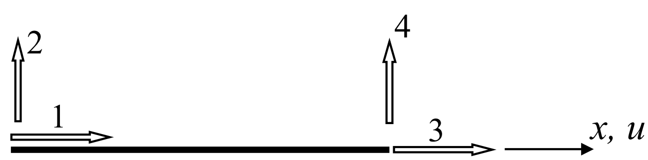

Figure 2: Truss element with four DOFs.