The Q4 Element

This document considers finite elements that carry load only in their plane. These elements are sometimes referred to as "2D continuum elements," but the term "plane elements" is preferred in these documents. The principle of virtual displacements is utilized to derive expressions for the stiffness matrix and load vector, commencing with the generic principle of virtual displacements, which reads

$$\int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_V \delta \tilde{\mathbf{u}}^T \mathbf{p}\, dV = 0 \tag{1}$$

where, in this document, **p** represents the distributed element loads:

$$\mathbf{p} = \begin{Bmatrix} p_x(x,y) \\ p_y(x,y) \end{Bmatrix} \tag{2}$$

The principle of virtual displacements does by itself represent an average satisfaction of equilibrium. Conversely, material law and kinematics must be added. To this end, the material law for plane problems is written $\boldsymbol{\sigma}=\mathbf{D}\boldsymbol{\varepsilon}$ and is either plane stress:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3}$$

or plane strain:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4}$$

Substitution of the material law into Eq. (1) yields

$$\int_V \delta \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon}\, dV - \int_V \delta \tilde{\mathbf{u}}^T \mathbf{p}\, dV = 0 \tag{5}$$

Next, the kinematic relationship

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \nabla \tilde{\mathbf{u}} \tag{6}$$

is combined with the fundamental finite element assumption

$$\tilde{\mathbf{u}} = \mathbf{N}\mathbf{u} \tag{7}$$

to read

$$\varepsilon = \nabla\tilde{\mathbf{u}} = \nabla\mathbf{N}\mathbf{u} \equiv \mathbf{B}\mathbf{u} \tag{8}$$

where the **B**-matrix is defined as $\mathbf{B} = \nabla\mathbf{N}$. It is understood that the **B**-matrix contains the derivative of the shape functions. Substitution of Eq. (8) into (5) with the same discretization of the virtual displacement field as the real one yields

$$\int_V (\mathbf{B}\delta\mathbf{u})^T \mathbf{D}(\mathbf{B}\mathbf{u})\,dV - \int_V (\mathbf{N}\delta\mathbf{u})^T \mathbf{p}\,dV = 0 \tag{9}$$

Because the transpose of a matrix product is the same as transposing each matrix or vector and switching the multiplication order, rearranging yields

$$\delta\mathbf{u}^T\left(\left(\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV\right)\mathbf{u} - \int_V \mathbf{N}^T\mathbf{p}\,dV\right) = 0 \tag{10}$$

The virtual displacement pattern is arbitrary, which means that the large parenthesis must be zero, and the result is:

$$\underbrace{\left(\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\,dV\right)}_{\mathbf{K}}\mathbf{u} = \underbrace{\int_V \mathbf{N}^T\mathbf{p}\,dV}_{\mathbf{F}} \tag{11}$$

where the stiffness matrix and load vector are identified. So far in this derivation the shape function relationship $\tilde{\mathbf{u}} = \mathbf{N}\mathbf{u}$ have remained abstract. The specific formulation of these shape functions for an arbitrary element is difficult because each element has a different shape and different dimensions. This problem is addressed next, by the isoparametric element formulation, which has two crucial steps:

1. The shape functions are formulated in a normalized coordinate system (ξ,η) instead of the original (x,y) coordinate system, and
2. The original shape of the element is described by means of the same shape functions that describe the deformation of the element. This likeness gives rise to the word *iso*, which means equal.

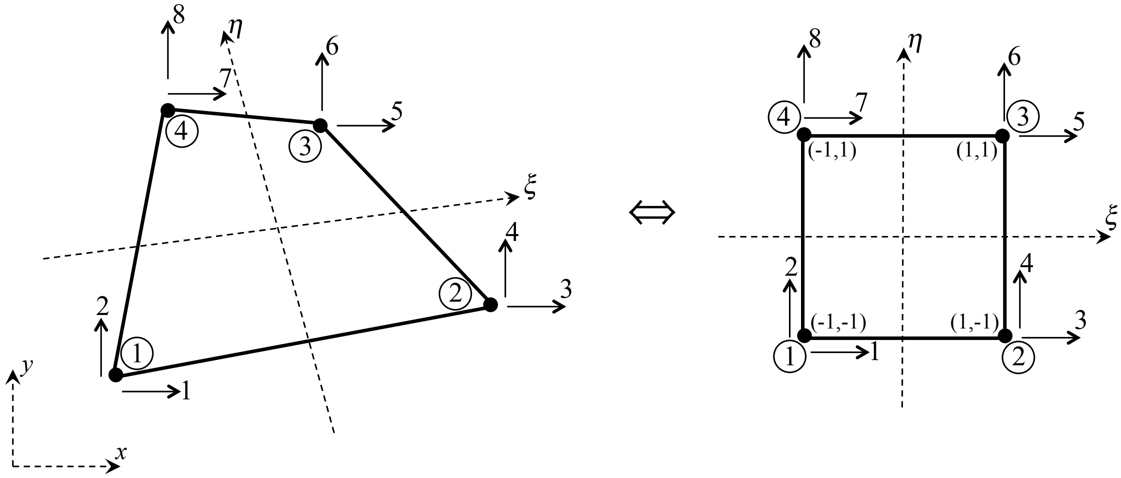

Figure 1: Transformation of the Quad4 element into the normalized domain.

Figure 1 shows an arbitrary element on the left-hand side, which is transformed into the normalized (ξ,η)-domain. All elements, regardless of shape, are equal in the (ξ,η)-domain. Hence, the problem of formulating shape functions is reduced to that of formulating shape functions for the right-most element in Figure 1. Each shape function must equal unity at the node that contains the DOF it is associated with. Furthermore, it must be zero at all other nodes. It is also natural to employ the same interpolation of the ξ-direction displacement as the η-direction displacement. In other words, it is natural to select the shape function of u_1 equal to that of u_2. Similarly, the shape functions of u_3 and u_4, u_5 and u_6, u_7 and u_8 are equal. Consequently, the discretization is written

$$\begin{Bmatrix} \tilde{u}(\xi,\eta) \\ \tilde{v}(\xi,\eta) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & N_3(\xi,\eta) & 0 & N_4(\xi,\eta) \end{array} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \quad (12)$$

where the shape functions N_1, N_2, N_3, and N_4 are associated with the four nodes of the element. Specifically, the following shape functions are derived

$$\begin{aligned} N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \quad (13)$$

Substitution of these shape function into Eq. (11) to calculate the stiffness matrix, **K**, and the load vector, **F**, poses a new problem: The integrals in Eq. Eq. (11) are formulated in the original (x,y)-domain, while the shape functions are formulated in the normalized (ξ,η)-domain. Coordinate transformation is necessary to remedy this problem. For pedagogical purposes, consider first the coordinate transformation of a single-fold integral:

$$\int_x f(x)\,dx = \int_\xi f(x(\xi))\frac{dx}{d\xi}\,d\xi \tag{14}$$

Notice that the factor dx/dξ appears in the integrand when the integral is calculated in the ξ-domain. Next, consider the transformation of a two-fold integral:

$$\int_y\int_x f(x,y)\,dx\,dy = \int_\eta\int_\xi f\big(x(\xi,\eta),y(\xi,\eta)\big)|\mathbf{J}|\,d\xi\,d\eta \tag{15}$$

where $|\mathbf{J}|$, sometimes denoted simply by J, is the determinant of the Jacobian matrix, that is established shortly with the following terms:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{16}$$

In summary, the integral for the stiffness matrix from Eq. (11) reads

$$\mathbf{K} = h\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}|\,d\xi\;d\eta \tag{17}$$

where the element thickness h is considered constant and pulled outside the integral. The load vector is addressed in a subsection below. The **B**-matrix must be computed for the evaluation of Eq. (17). According to Eq. (8) it contains derivatives of the shape functions. Specifically, the derivative operator in Eq. (6) applied to the matrix of shape functions in Eq. (12) gives

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \left[\begin{array}{cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \tag{18}$$

However, while the differentiation is with respect to the (x,y)-coordinates, the shape functions, N_i, are defined in terms of the (ξ,η)-coordinates. Suppose a relationship between the two coordinate systems is available, such that the coordinates of one system are expressed in terms of the other coordinates, i.e., $\xi(x,y)$, $\eta(x,y)$, $x(\xi,\eta)$, and $y(\xi,\eta)$. Then the sought derivatives are obtained by the chain rule of differentiation:

$$\frac{\partial N_i\left(\xi(x,y),\eta(x,y)\right)}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{19}$$

and

$$\frac{\partial N_i\left(\xi(x,y),\eta(x,y)\right)}{\partial y} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{20}$$

In these expressions the derivatives $\partial N_i/\partial \xi$ and $\partial N_i/\partial \eta$ are readily computed by hand. The derivatives $\partial \xi/\partial x$, etc. are picked from the inverse of the Jacobian matrix:

$$\mathbf{J}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \tag{21}$$

The remaining question is how to calculate the Jacobian matrix, i.e., the matrix of derivatives of original coordinates with respect to the normalized coordinates. This is at the heart of the isoparametric element formulation. In this approach, the coordinates of any point in the element, denoted $\tilde{\mathbf{x}}$, is expressed in terms of the nodal coordinates, which are collected in the vector **x**:

$$\tilde{\mathbf{x}} = \mathbf{N}\mathbf{x} \tag{22}$$

This equation has exactly the same form as Eq. (12) and **N** are the same shape functions, hence the phrase isoparametric. Eq. (22) is written as two equations in index notation as follows:

$$\begin{aligned} x &= N_i x_i \\ y &= N_i y_i \end{aligned} \tag{23}$$

where x and y identify the location of any point in the element and summation over equal indices is implied. Given this isoparametric element formulation the derivatives in the Jacobian matrix are

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \sum \frac{\partial N_i}{\partial \xi} x_i & \qquad \frac{\partial x}{\partial \eta} &= \sum \frac{\partial N_i}{\partial \eta} x_i \\ \frac{\partial y}{\partial \xi} &= \sum \frac{\partial N_i}{\partial \xi} y_i & \qquad \frac{\partial y}{\partial \eta} &= \sum \frac{\partial N_i}{\partial \eta} y_i \end{aligned} \tag{24}$$

This concludes the evaluation of the integrand, but not the integral, in the stiffness matrix expression in Eq. (17). For arbitrary element shapes it is not possible to evaluate the integral analytically. Rather, numerical integration is necessary.

Quadrature

Several numerical integration schemes, called quarature, are available to evaluate the integral in Eq. (17) in an approximate manner. The most common quadrature for four-node quadrilateral elements is Gauss integration. Consider first Gauss quadrature for single-fold integration of the arbitrary function $f(\xi)$ over the domain -1 to 1:

$$\int_{-1}^{1} f(\xi)\ d\xi = \sum_{i=1}^{N} w_i \cdot f(\xi_i) \tag{25}$$

where N is the number of integration points, w_i are integration weights, and ξ_i are integration points. More details about quadrature are written in a math document on the topic. For example, for N=1 the rule gives w_1=2 and ξ_1=0. For N=2, Gauss quadrature specifies w_1=w_2=1, $\xi_1\approx-0.577$, and $\xi_2\approx0.577$. Gauss quadrature is readily extended to two-fold integrals by applying the same points and weights in both directions. For example, two-point Gauss quadrataure in the (ξ,η)-domain evaluates the integrand at the four points (0.577, 0.577), (-0.577, 0.577), (0.577, -0.577), and (-0.577, -0.577) with unit weight associated with each point. In conclusion, the stiffness matrix integral in Eq. (17) is evaluated by

$$\mathbf{K} = h\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\ \mathbf{B}|\mathbf{J}|\,\mathrm{d}\xi\, d\eta = h\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \left(\mathbf{B}^T\mathbf{D}\ \mathbf{B}\ |\mathbf{J}|\right)_{ij} \tag{26}$$

It is noted that all the information about the element's geometry is contained in **J**.

Mechanisms

This element has the following eigenmodes, i.e., independent displacement modes:

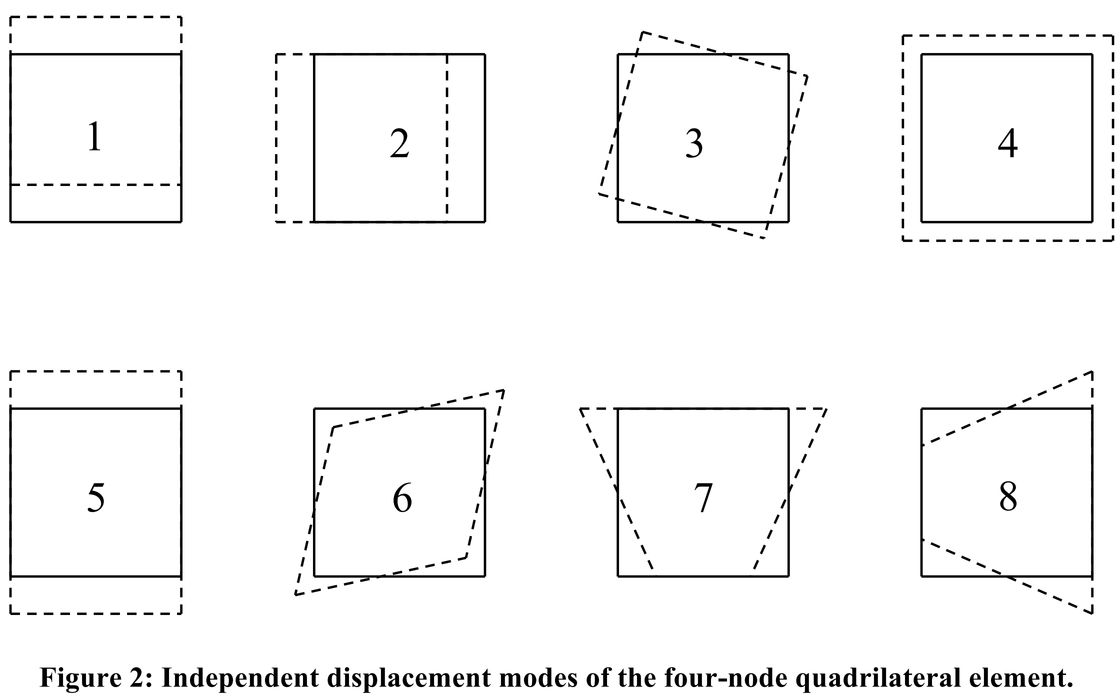

Figure 2: Independent displacement modes of the four-node quadrilateral element.

Consistent Load Vector

In finite element analysis it is not generally possible to lump distributed element loads as concentrated nodal loads according to simple tributary area. Rather, the load vector as defined in Eq. (11) must be evaluated. Evaluation of the load vector according to that expression, which contains the shape functions of the element formulation, results in what is called the consistent load vector. One useful case is load that acts on the element edge. The load acts in the element plane, and usually has both an *x* and a *y*-component. That is, both components of the vector **p**, as defined in Eq. (2), are generally non-zero. In the following derivations, let the coordinate Γ run along the loaded edge in the original (*x*,*y*)-coordinate system. In the normalized coordinate system, either ξ or η runs long the edge, depending on which side of the element is loaded. Without loss of generality, consider an example where the edge ξ=1 is loaded. The coordinate η run along this edge. The expression for the load vector reads

$$\int_V \mathbf{N}^T \mathbf{p}\, dV = \int_\Gamma \mathbf{N}^T \mathbf{p}\, d\Gamma = \int_{-1}^{1} \mathbf{N}^T \mathbf{p} \frac{d\Gamma}{d\eta} d\eta \tag{27}$$

The value of the Jacobian, dΓ/dη, depends on the orientation of the loaded edge. An infinitesimal length dΓ along the edge is related to infinitesimal lengths along the coordinate directions, *dx* and *dy*, by

$$d\Gamma^2 = dx^2 + dy^2 \tag{28}$$

Furthermore, because ξ is constant along the edge, the differentials *dx* and *dy* are related to the differentials in the normalized coordinate system by

$$\begin{aligned} dx &= \frac{dx}{d\eta} d\eta \\ dy &= \frac{dy}{d\eta} d\eta \end{aligned} \tag{29}$$

Combination of Eqs. (28) and (29) yields

$$d\Gamma = d\eta \cdot \sqrt{\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2} \tag{30}$$

where the derivatives from the Jacobian matrix are evaluated at the edge ξ=1.