The M4 Element

A variety of plate elements exist, some being characterized as Kirchhoff elements, i.e., for thin plates, and others as Mindlin elements, i.e., for thick plates. The early developments focused on triangular elements but a quadrilateral Mindlin element with bilinear shape functions is shown in Figure 1 and derived below.

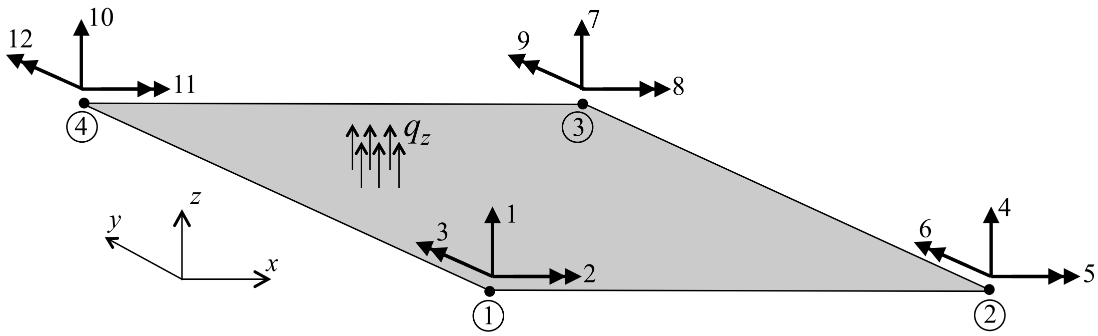

Figure 1: Bilinear Mindlin element.

As a starting point, consider the principle of virtual displacements:

$$\underbrace{\int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV}_{\delta W_{int}} - \underbrace{\int_A q_z \cdot \delta w \, dA}_{\delta W_{ext}} = 0 \tag{1}$$

where, in the Mindlin formulation, the stress and strain vectors contain everything except the *zz*-components:

$$\boldsymbol{\varepsilon} = \{\underbrace{\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}}_{\boldsymbol{\varepsilon}_b}, \underbrace{\gamma_{yz}, \gamma_{zx}}_{\boldsymbol{\varepsilon}_b}\} \tag{2}$$

$$\boldsymbol{\sigma} = \{\underbrace{\sigma_{xx}, \sigma_{yy}, \tau_{xy}}_{\boldsymbol{\sigma}_b}, \underbrace{\tau_{yz}, \tau_{zx}}_{\boldsymbol{\sigma}_s}\} \tag{3}$$

The three first components of the stress and strain vectors are associated with bending; hence, they are collected in vectors with the subscript *b*. The relationship between those stress and strain components are governed by the plain stress material law, which is appropriate when the plate is not unreasonably thick:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\sigma}_b = \mathbf{D}_b \boldsymbol{\varepsilon}_b \tag{4}$$

The last two stress and strain components, labelled with the subscript *s*, are related to shear deformation and are independent of the other components, related by the shear modulus $G=E/(2(1+\nu))$:

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\sigma}_s = \mathbf{D}_s \boldsymbol{\varepsilon}_s \tag{5}$$

Substitution of Eqs. (4) and (5) into Eq. (1) yields the following two-part expression for the internal virtual work

$$\begin{aligned} \delta W_{int} &= \int_V \delta \boldsymbol{\varepsilon}_b^T \boldsymbol{\sigma}_b \, dV + \int_V \delta \boldsymbol{\varepsilon}_s^T \boldsymbol{\sigma}_s \, dV \\ &= \int_V \delta \boldsymbol{\varepsilon}_b^T \mathbf{D}_b \boldsymbol{\varepsilon}_b \, dV + \int_V \delta \boldsymbol{\varepsilon}_s^T \mathbf{D}_s \boldsymbol{\varepsilon}_s \, dV \end{aligned} \tag{6}$$

Next, the kinematic equations from Mindlin plate theory are written in matrix form, maintaining the separation between the bending-related and shear-related parts:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z \cdot \begin{Bmatrix} -\theta_{y,x} \\ \theta_{x,y} \\ \theta_{x,x} - \theta_{y,y} \end{Bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\varepsilon}_b = -z \cdot \boldsymbol{\kappa}_b \tag{7}$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} -\theta_x + w_{,y} \\ \theta_y + w_{,x} \end{Bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\varepsilon}_s = \boldsymbol{\kappa}_s \tag{8}$$

where $\boldsymbol{\kappa}_b$ and $\boldsymbol{\kappa}_s$ are loosely referred to as curvature vectors for bending and shear, respectively. Substitution of Eqs. (7) and (8) into Eq. (1) yields

$$\delta W_{int} = \int_V z^2 \cdot \delta \boldsymbol{\kappa}_b^T \mathbf{D}_b \boldsymbol{\kappa}_b \, dV + \int_V \delta \boldsymbol{\kappa}_s^T \mathbf{D}_s \boldsymbol{\kappa}_s \, dV \tag{9}$$

Those volume integrals are now transformed into area integrals by carrying out the integration with respect to z from $-h/2$ to $h/2$. Because all the components of the curvature vectors are constant with respect to z, the result is:

$$\delta W_{int} = \frac{h^3}{12} \cdot \int_A \delta \boldsymbol{\kappa}_b^T \mathbf{D}_b \boldsymbol{\kappa}_b \, dA + h \cdot \int_A \delta \boldsymbol{\kappa}_s^T \mathbf{D}_s \boldsymbol{\kappa}_s \, dA \tag{10}$$

This can also be expressed in terms of modified **D**-matrices that include the plate thickness:

$$\delta W_{int} = \int_A \delta \boldsymbol{\kappa}_b^T \tilde{\mathbf{D}}_b \boldsymbol{\kappa}_b \, dA + \int_A \delta \boldsymbol{\kappa}_s^T \tilde{\mathbf{D}}_s \boldsymbol{\kappa}_s \, dA \tag{11}$$

where

$$\tilde{\mathbf{D}}_b = \frac{Eh^3}{12(1-\nu^2)} \cdot \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{12}$$

and

$$\tilde{\mathbf{D}}_s = \begin{bmatrix} Gh & 0 \\ 0 & Gh \end{bmatrix} \tag{13}$$

Having substituted material law and kinematics into the internal virtual work in Eq. (1), the discretization of the problem by shape functions is addressed. For Mindlin elements the displacement field, w, as well as the two rotation fields, θ_x and θ_y, are discretized. For the four-node element shown Figure 1, the following bilinear shape functions are used:

$$\begin{aligned} N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{14}$$

With reference to the degrees of freedom in Figure 1, the complete discretization reads:

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \left[\begin{array}{ccc:ccc:ccc:ccc} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{Bmatrix} \tag{15}$$

In short, the discretization reads

$$\tilde{\mathbf{u}} = \mathbf{N}\mathbf{u} \tag{16}$$

It can also be spelled out as

$$w = N_1 u_1 + N_2 u_4 + N_3 u_7 + N_4 u_{10} \tag{17}$$

$$\theta_x = N_1 u_2 + N_2 u_5 + N_3 u_8 + N_4 u_{11} \tag{18}$$

$$\theta_y = N_1 u_3 + N_2 u_6 + N_3 u_9 + N_4 u_{12} \tag{19}$$

This discretization is substituted into the weak form of the boundary value problem in Eq. (11) via the curvature vectors, which in turn are expressed in terms of the shape functions by the **B**-matrices:

$$\begin{aligned}
\boldsymbol{\kappa}_b &= \begin{Bmatrix} -\theta_{y,x} \\ \theta_{x,y} \\ \theta_{x,x} - \theta_{y,y} \end{Bmatrix} \\
&= \begin{Bmatrix} -N_{1,x}u_3 - N_{2,x}u_6 - N_{3,x}u_9 - N_{4,x}u_{12} \\ N_{1,y}u_2 + N_{2,y}u_5 + N_{3,y}u_8 + N_{4,y}u_{11} \\ N_{1,x}u_2 + N_{2,x}u_5 + N_{3,x}u_8 + N_{4,x}u_{11} - N_{1,y}u_3 - N_{2,y}u_6 - N_{3,y}u_9 - N_{4,y}u_{12} \end{Bmatrix} \\
&= \left[\begin{array}{ccc:ccc:ccc:ccc} 0 & 0 & -N_{1,x} & 0 & 0 & -N_{2,x} & 0 & 0 & -N_{3,x} & 0 & 0 & -N_{4,x} \\ 0 & N_{1,y} & 0 & 0 & N_{2,y} & 0 & 0 & N_{3,y} & 0 & 0 & N_{4,y} & 0 \\ 0 & N_{1,x} & -N_{1,y} & 0 & N_{2,x} & -N_{2,y} & 0 & N_{3,x} & -N_{3,y} & 0 & N_{4,x} & -N_{4,y} \end{array}\right]\mathbf{u} \\
&= \mathbf{B}_b\mathbf{u}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\boldsymbol{\kappa}_s &= \begin{Bmatrix} -\theta_x + w_{,y} \\ \theta_y + w_{,x} \end{Bmatrix} \\
&= \begin{Bmatrix} -N_1u_2 - N_2u_5 - N_3u_8 - N_4u_{11} + N_{1,y}u_1 + N_{2,y}u_4 + N_{3,y}u_7 + N_{4,y}u_{10} \\ N_1u_3 + N_2u_6 + N_3u_9 + N_4u_{12} + N_{1,x}u_1 + N_{2,x}u_4 + N_{3,x}u_7 + N_{4,x}u_{10} \end{Bmatrix} \\
&= \left[\begin{array}{ccc:ccc:ccc:ccc} N_{1,y} & -N_1 & 0 & N_{2,y} & -N_2 & 0 & N_{3,y} & -N_3 & 0 & N_{4,y} & -N_4 & 0 \\ N_{1,x} & 0 & N_1 & N_{2,x} & 0 & N_2 & N_{3,x} & 0 & N_3 & N_{4,x} & 0 & N_4 \end{array}\right]\mathbf{u} \\
&= \mathbf{B}_s\mathbf{u}
\end{aligned} \tag{21}$$

Substitution of Eqs. (20) and (21) into Eq. (11), which is now combined with Eq. (1) yields

$$\int_A (\mathbf{B}_b\,\delta\mathbf{u})^T\,\tilde{\mathbf{D}}_b(\mathbf{B}_b\mathbf{u})\,dA + \int_A (\mathbf{B}_s\,\delta\mathbf{u})^T\,\tilde{\mathbf{D}}_s(\mathbf{B}_s\mathbf{u})\,dV - \int_A q_z\cdot(\mathbf{N}_w\,\delta\mathbf{u})\,dA = 0 \tag{22}$$

where

$$\mathbf{N}_w = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \end{bmatrix} \tag{23}$$

Rearranging Eq. (22) yields

$$\delta\mathbf{u}^T\left(\left[\int_A \mathbf{B}_b^T\tilde{\mathbf{D}}_b\mathbf{B}_b\,dA + \int_A \mathbf{B}_s^T\tilde{\mathbf{D}}_s\mathbf{B}_s\,dV\right]\mathbf{u} - \int_A q_z\cdot\mathbf{N}_w^T\,dA\right) = 0 \tag{24}$$

For arbitrary virtual displacements $\delta\mathbf{u}$ the parenthesis must be zero, hence

$$\underbrace{\left[\int_A \mathbf{B}_b^T \tilde{\mathbf{D}}_b \mathbf{B}_b \, dA + \int_A \mathbf{B}_s^T \tilde{\mathbf{D}}_s \mathbf{B}_s \, dV\right]}_{\mathbf{K}} \mathbf{u} = \underbrace{\int_A q_z \cdot \mathbf{N}_w^T \, dA}_{\mathbf{F}} \qquad (25)$$

where the stiffness matrix and load vector have been identified.