System Reliability

This material was first described to me in a course taught by Professor Armen Der Kiureghian at the University of California at Berkeley. In 2005 he made an excellent description available in Chapter 14 "First- and second-order reliability methods" of the CRC Engineering Design Reliability Handbook edited by Nikolaidis, Ghiocel and Singhal, published by the CRC Press in Boca Raton, Florida.

System reliability analysis deals with problems where failure is defined in terms of more than one limit-state function. One example is a structural element with two failure modes, say, shear and bending. Suppose one limit-state function is defined for shear failure and another for bending failure. The failure event is if any of the two limit-state functions are negative. Thus, this is an example of a system reliability problem. Conversely, reliability problems with only one limit-state function are called component reliability problems.

Formulation of System Reliability Problems

For reference, the component reliability problem, i.e., the problem when only one limit-state function is involved, reads

$$p_f = P\left(g(\mathbf{x}) \leq 0\right) \tag{1}$$

In contrast, a series system reliability problem is characterized by failure occurring if any of several limit-state functions are negative:

$$p_f = P\left(\bigcup_{k=1}^{K}\left(g_k(\mathbf{x}) \leq 0\right)\right) \tag{2}$$

where K is the number of limit-state functions. Another special case of system reliability problems is the parallel system, in which failure occurs only when all the limit-state functions are negative:

$$p_f = P\left(\bigcap_{k=1}^{K}\left(g_k(\mathbf{x}) \leq 0\right)\right) \tag{3}$$

However, neither the series system nor the parallel system describes the general system reliability problem. Rather the general system reliability problem is formulated as a series system of parallel systems:

$$p_f = P\left(\bigcup_{m=1}^{M}\bigcap_{j\in c_m}\left(g_j(\mathbf{x}) \leq 0\right)\right) \tag{4}$$

where M is the number of parallel systems and c_m is the number of so-called cut sets. The terminology is further developed in the field of operations research. In the context of reliability analysis, cut set is another name for a sub parallel system. The logical operator in Eq. (4) $j \in c_m$ means that only limit-state functions within cut set number m is included. The cut set formulation in Eq. (4) can be replaced by the equally general link set formulation, in which the problem is defined as a parallel system of sub series systems. However, this is uncommon and less efficient in structural reliability analysis. To this end, unless the problem at hand is a pure series or parallel system, it is an important task to identify the cut sets of the problem. Three types exist: general cut sets, minimum cut sets, and disjoint cut sets. The following step-wise procedure is suggested:

1. **Identify all cut sets:** Identify all failure scenarios, i.e., all combinations of component failure that would cause system failure. A general cut set is any set of components whose joint failure constitutes failure of the system.
2. **Identify the minimum cut sets:** A minimum cut set contains the minimum number of components; if any component is removed from the set then it ceases to be a cut set. Usually, when care is exercised in Step 1 then the minimum cut sets are already specified in that step.
3. **If possible, identify the disjoint cut sets:** By definition, two cut sets are disjoint when their intersection is the null event. In other words, if two cut sets are disjoint then they cannot occur simultaneously. An important consequence of this is that the failure probability for a general system with disjoint cut sets is that the system failure probability is simply the sum of the cut set failure probabilities. However, it is rarely efficient to pursue disjoint cut sets for efficiency reasons. The reason is that to establish disjoint cut sets it is necessary to include the complement of component failure events in the cut sets. This increases the number of components in the cut sets, which increases the cost of computing their failure probability.

Reliability Block Diagrams

Reliability block diagrams (RBDs) are sometimes helpful to visualize system reliability problems and identify cut sets. An RBD is a sketch that connects the components in the same way as a road map. Figure 1 shows an example of a simple RBD. The top of the figure shows the physical problem, namely the road network between two cities. The RBD is shown at the bottom. For road network problems the sketch of the physical problem is particularly close to the RBD. However, the direct mapping of the physical problem into an RBD is not generally possible. The road network problem is utilized here for pedagogical reasons because any RBD is best interpreted as a road network between two cities. This interpretation helps when identifying cut sets; the failure of any set of components that would close the connection between the two cities is a cut set. The road network analogy is why it is more straightforward to identify cut sets after the RBD has been sketched. For the example in Figure 1 it is relatively easy to see that failure of Tunnel 2 is one cut set, and the simultaneous failure of the bridge and Tunnel 1 is another cut set. The failure of any of these two cut sets will sever the connection between the two cities. Hence, the system failure event is written $BT_1 \cup T_2$.

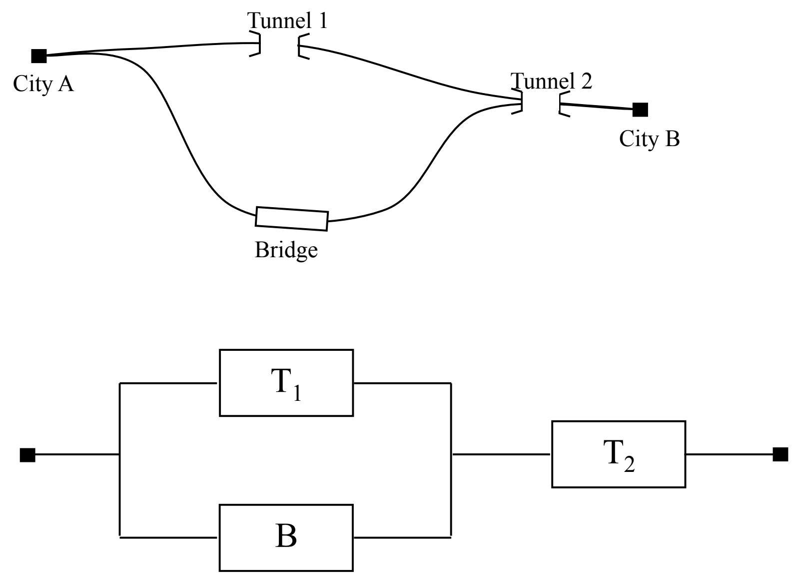

Figure 1: Reliability block diagram.

FORM Solution for Series and Parallel Problems

The series system reliability problem is written in Eq. (2). It is equivalently written in the standard normal space:

$$p_f = P\left(\bigcup_{k=1}^{K}\left(G_k(\mathbf{y}) \leq 0\right)\right) \tag{5}$$

where $\mathbf{y}$ is the vector of standard normal random variables. Linearizing each limit-state function according to FORM component reliability analysis yields:

$$p_f = P\left(\bigcup_{k=1}^{K}\left(\beta_k - \boldsymbol{\alpha}_k^T \mathbf{y} \leq 0\right)\right) \tag{6}$$

where β_k and $\boldsymbol{\alpha}_k$ are, respectively, the reliability index and alpha-vector for each limit-state function. For each limit-state function, the random variable

$$z_k = \boldsymbol{\alpha}_k^T \mathbf{y} \tag{7}$$

is now defined. According to the joint uncorrelated standard normal distribution for $\mathbf{y}$ the z_k variables have zero mean. Also, because the alpha-vectors are normalized the z_k variables have unit variances. The covariance between z_i and z_j is

$$\text{Cov}\left[\boldsymbol{\alpha}_i^T \mathbf{y}, \boldsymbol{\alpha}_j^T \mathbf{y}\right] = \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_{yy} \boldsymbol{\alpha}_j = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j \tag{8}$$

i.e., the dot product of the two alpha-vectors. This covariance is equal to the correlation because the z_k variables have unit standard deviation. Substitution of Eq. (7) into Eq. (6) yields the following expression for the system failure probability:

$$p_f = P\left(\bigcup_{k=1}^{K}\left(\beta_k \leq z_k\right)\right) \tag{9}$$

To arrive at an expression involving the intersection operator rather than the union operator, the complementary probability rule is first invoked:

$$p_f = 1 - P\left(\overline{\bigcup_{k=1}^{K} (\beta_k \le z_k)}\right) \tag{10}$$

followed by the application of one of de Morgan's rules:

$$p_f = 1 - P\left(\bigcap_{k=1}^{K} \overline{(\beta_k \le z_k)}\right) = 1 - P\left(\bigcap_{k=1}^{K} (z_k < \beta_k)\right) \tag{11}$$

Because z_k are correlated standard normal random variables the k-dimensional joint standard normal CDF is identified:

$$p_f = 1 - \Phi_K(\boldsymbol{\beta}, \mathbf{R}) \tag{12}$$

where $\boldsymbol{\beta}$ is the vector of reliability indices for all the limit-state functions and $\mathbf{R}$ is the correlation matrix for the random variables z_k. The correlation matrix contains the covariances in Eq. (8) because the standard deviations are equal to one. Unfortunately, it is difficult to evaluate the joint CDF in Eq. (12). The development of accurate approximations remains an open research area. For k=2 the following analytical solution is available:

$$\Phi_2(-\boldsymbol{\beta}, \mathbf{R}) = \Phi(-\beta_1) \cdot \Phi(-\beta_2) + \int_0^{\rho_{12}} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{\beta_1^2 + \beta_2^2 - 2 \cdot \rho \cdot \beta_1 \cdot \beta_2}{2 \cdot (1-\rho^2)}\right] d\rho \tag{13}$$

where the last term is the integral of the bivariate standard normal PDF with correlation, and the correlation between the two limit-states according to Eq. (8) is

$$\rho_{12} = \boldsymbol{\alpha}_1^T \boldsymbol{\alpha}_2 \tag{14}$$

It is noted that the correlation coefficient must exceed, say, 0.5 before the intersection probability becomes significant. In Eq. (13) it is also observed that the joint CDF is equal to the product of the two component failure probabilities when the two failure modes are uncorrelated. Otherwise an integral must be evaluated to obtain the joint failure probability. Carrying out the derivations above for parallel systems yields:

$$p_f = \Phi_K(-\boldsymbol{\beta}, \mathbf{R}) \tag{15}$$

Sampling Analysis

Although sampling analysis is usually computationally expensive it is appealingly straightforward to implement for system reliability problems. For each sample is it checked whether the any of the limit-states are violated and if this caused system failure. Depending on the result the indicator variable is set to zero or one and another sample is generated. Hence, for limit-state functions that are computationally inexpensive to evaluate, mean-centred sampling is a straightforward alternative to obtain an estimate of the failure probability.

System Reliability Bounds

The difficulties associated with evaluating the system failure probability even for series and parallel systems have led to the exploration of probability bounds. For series systems there are uni-modal, bi-modal, and tri-modal bounds. These three options are understood by considering the inclusion-exclusion rule of probability for the series system problem:

$$p_f = P\left(\bigcup_{m=1}^{M} C_m\right) = \sum_{m=1}^{M} P(C_m) - \sum_{\substack{m=1 \\ i<m}}^{M} P(C_m C_i) + \sum_{\substack{m=1 \\ j<i<m}}^{M} P(C_m C_i C_j) - \cdots \qquad (16)$$

where C_m denote the event that limit-state number m is violated, or more generally the event that cut set number m is in the failure state. Uni-modal bounds are formulated by retaining only the first term in Eq. (16):

$$\max(p_m) \quad \le \quad p_f \quad \le \quad \min\left(1, \sum_{m=1}^{M} p_m\right) \qquad (17)$$

where the shorthand notation $p_m = P(C_m)$ is introduced. Bi-modal bounds is an extension where also the two-component joint probabilities are included:

$$p_1 + \sum_{m=2}^{M} \max\left(0, p_m - \sum_{j=1}^{m-1} p_{mj}\right) \quad \le \quad p_f \quad \le \quad \mathrm{p}_1 + \sum_{m=2}^{M}\left(p_m - \max_{j<m}(p_{mj})\right) \qquad (18)$$

where the shorthand notation $p_{ij} = P(C_i C_j)$ is introduced and evaluated according to Eq. (13). To make the bi-modal bounds more accessible they are here illustrated for up to four components:

$$\begin{array}{l} p_1 + \max\big(0, p_2 - p_{21}\big) \\ + \max\big(0, p_3 - (p_{31} + p_{32})\big) \\ + \max\big(0, p_4 - (p_{41} + p_{42} + p_{43})\big) \end{array} \quad \le \quad p_f \quad \le \quad \begin{array}{l} p_1 + p_2 - p_{21} \\ + p_3 - \max(p_{31}, p_{32}) \\ + p_4 - \max(p_{41}, p_{42}, p_{43}) \end{array} \qquad (19)$$

In theory the bounds will vary somewhat depending on the ordering of the limit-state functions. Hence, all combinations should be checked to ensure that the bounds are not falsely narrow. Beyond uni-modal and bi-modal bounds, tri-modal bounds are also formulated in the literature in terms of $p_{ijk} = P(C_i C_j C_k)$ but not covered here because of the additional complexity in evaluating p_{ijk}.