Stochastic Fatigue

The objective in this document is to estimate the fatigue life of a structure, when the loading is a continuous stochastic process. To address this problem, the concepts of S-N curves and Miner's rule from the document on deterministic fatigue are applied in the context of a stochastic stress history. This helps simplify the problem, but it is still a challenging one without general closed-form solutions.

Rayleigh Approximation

According to Miner's rule, fatigue damage is measured by D and defined by

$$D = \sum_{i=1}^{B} \frac{n_i}{N_i} \tag{1}$$

which steadily increases from zero to unity, at which failure occurs. In Eq. (1), B is the number of bins or stress ranges, n_i is the number of cycles in each stress range, and N_i is the number of cycles that causes failure in in that range. In stochastic fatigue it is helpful to think of a damage-increment, ΔD, which adds to the damage in each stress cycle. Of course, each cycle may have a different damage-increment, but for a given stress range it is assumed that ΔD is the same in each cycle. In other words, the damage-increment in a cycle is uniquely defined by the stress range of that cycle. According to Eq. (1), the damage-increment in each cycle is

$$\Delta D_i = \frac{1}{N_i} \tag{2}$$

where ΔD_i and N_i are the damage-increment and number of cycles to failure at stress range number i. With that definition of ΔD_i it is possible to deal with situations with different stress range in each cycle. In that case there are as many bins as the number of cycles, and failure occurs when

$$D = \sum_{i=1}^{N_f} \Delta D_i = 1 \tag{3}$$

where N_f is the total number of stress cycles to failure, each cycle with its own stress range. Eq. (3) contains a sought quantity, namely N_f, a measure of the fatigue life of the structure. To estimate this quantity, the expectation of Eq. (3) is considered:

$$E[D] = \sum_{i=1}^{N_f} E[\Delta D] = E[N_f] \cdot E[\Delta D] = 1 \tag{4}$$

Solving for E[N_f] yields

$$E[N_f] = \frac{1}{E[\Delta D]} \tag{5}$$

From the total expected number of cycles, E[N_f], the expected fatigue life is obtained by multiplying by the duration of each cycle. The average duration of the cycles is the inverse of the average rate of occurrence of cycles. For a Gaussian process that rate is approximated by the up-crossings of the mean stress, which is

$$\nu_U^+(\mu_U) = \frac{1}{2\pi} \cdot \sqrt{\frac{\lambda_2}{\lambda_0}} \tag{6}$$

As a result, the fatigue life estimate is the expected number of cycles multiplied by the average cycle duration:

$$E[T] = E[N_f] \cdot \frac{1}{\nu_U^+(\mu_U)} = \frac{2\pi}{E[\Delta D]} \cdot \sqrt{\frac{\lambda_0}{\lambda_2}} \tag{7}$$

The remaining task is to address the damage-increment. To link the amplitude of a stress cycle to a particular damage increment, the generic expression for an S-N curve (those are obtained from experimental data and introduce the "capacity" of the material) is substituted into Eq. (2):

$$\Delta D_i = \frac{1}{N_i} = \frac{1}{\left(\frac{K}{S_i^m}\right)} = \frac{S_i^m}{K} \tag{8}$$

The expectation, needed in Eq. (7), is

$$E[\Delta D] = \frac{E[S^m]}{K} \tag{9}$$

In what is referred to as the Rayleigh approximation of stochastic fatigue analysis, it is here assumed that the stress peaks are Rayleigh distributed, which is appropriate for Gaussian processes. As a result, the m^{th} moment of the stress range, estimated as two times the peak, u_p, is

$$\begin{aligned} E[S^m] = E[(2u_p)^m] &= \int_0^\infty (2u_p)^m \cdot f_{u_p}(u_p)\, du_p \\ &= 2^m \cdot \int_0^\infty \frac{u_p^{m+1}}{\sigma_U^2} \cdot \exp\left(-\frac{u_p^2}{2\cdot\sigma_U^2}\right) du_p \\ &= 2^{1.5m} \cdot \sigma_U^m \cdot \Gamma\left(1+\frac{m}{2}\right) \\ &= 2^{1.5m} \cdot \lambda_0^{0.5m} \cdot \Gamma\left(1+\frac{m}{2}\right) \end{aligned} \tag{10}$$

In summary, the expected fatigue life is

$$E[T] = \frac{2\pi \cdot K \cdot 2^{-1.5m} \cdot \lambda_0^{0.5\cdot(1-m)} \cdot \lambda_2^{-0.5}}{\Gamma\left(1+\frac{m}{2}\right)} \tag{11}$$

Rainflow Analysis

The rainflow method for counting half-cycles is described in the document on deterministic fatigue considerations. One brute-force application of this method to stochastic fatigue is to simulate realizations of the stochastic stress process, followed by rainflow counting. Like Monte Carlo sampling, that approach is straightforward but accurate results come at a high computational cost. A special case that facilitates an analytical solution is when the S-N curve is described by *m*=1 (Lutes and Sarkani 1997). In that case, the damage increments are simply

$$\Delta D_i = \frac{S_i}{K} \tag{12}$$

As a result, the failure criterion is

$$D = \sum_{i=1}^{N_f} \Delta D_i = \frac{1}{K} \cdot \sum_{i=1}^{N_f} S_i = 1 \tag{13}$$

In this case, the rainflow sum of stress ranges is obtained by adding contributions from each infinitesimal time increment *dt*. Denoting the stress process by *U*(*t*), the increment of stress during *dt* is $\dot{U}(t) \cdot dt$. Each of these stress increments is part of a stress range counted by the rainflow method. Because the stress increment can be positive or negative, the absolute value is introduced to obtain the sum:

$$\sum_{i=1}^{N_f} S_i = \frac{1}{2} \cdot \int_0^T \left|\dot{U}(t)\right| dt \tag{14}$$

where the factor ½ is introduced because an actual full stress cycle consists of *two* increment paths, i.e., one towards more tension and the other towards more compression. The expected value of the integral is

$$E\left[\int_0^T \left|\dot{U}(t)\right| dt\right] = E[T] \cdot E\left[\left|\dot{U}(t)\right|\right] \tag{15}$$

Substitution of the expectation on the left-hand side of Eq. (15) for the integral in Eq. (14) and substitution of the sum in Eq. (14) into Eq. (13) yields:

$$E[T] = \frac{2K}{E\left[\left|\dot{U}(t)\right|\right]} \tag{16}$$

For a Gaussian process, the derivative process has the normal distribution

$$f_{\dot{U}}(u) = \frac{1}{\sqrt{2\pi} \cdot \sigma_{\dot{U}}} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{u}{\sigma_{\dot{U}}}\right)^2\right) \tag{17}$$

which means that the expectation of the absolute value in Eq. (16) is

$$E\left[\left|\dot{U}(t)\right|\right] = \int_0^\infty u \cdot f_{\dot{U}}(u)\,du = \sqrt{\frac{2}{\pi}} \cdot \sigma_{\dot{U}} \tag{18}$$

Substitution of Eq. (18) into Eq. (16) yields

$$E[T] = \frac{\sqrt{2\pi} \cdot K}{\sigma_{\dot{U}}} \tag{19}$$

This result can be compared with the Rayleigh approximation presented earlier by setting *m*=1 and $\lambda_2 = \sigma_{\dot{U}}^2$ in Eq. (11), which yields

$$E[T] = \frac{2\pi \cdot 2^{-1.5} \cdot K}{\Gamma(1.5) \cdot \sigma_{\dot{U}}} = \frac{\sqrt{2\pi} \cdot K}{\sigma_{\dot{U}}} \tag{20}$$

This coincidence of the rainflow approach with the Rayleigh approximation is due to the selection of *m*=1 in the S-N curve, but did not include any assumption on the bandwidth.

References

Lutes, L. D., and Sarkani, S. (1997). *Stochastic analysis of structural and mechanical vibrations*. Prentice Hall.