Stiffness Method

In modern structural analysis the stiffness method is by far the most central method. It is the basis for most structural analysis software, and it represents one avenue to the "finite element method," which is at the pinnacle of structural analysis tools. In the same way as the flexibility method is the quintessential force method, the stiffness method is the primary displacement method. In the stiffness method, equilibrium equations are established and solved for unknown joint displacements and rotations. The joints are usually referred to as nodes, and the unknowns are called degrees of freedom (DOFs).

The stiffness method is popular because it is easily implemented on the computer. No subjective choices are made, such as the choice of redundants in the flexibility methods. Rather, the computer straightforwardly assigns a pre-defined number of DOFs to each node and establishes a linear system of equilibrium equations automatically.

This document introduces the classical stiffness method, which is suitable for hand calculations. The classical approach is also useful as a pedagogical step before diving into other documents on the computational version of the stiffness method and the full-blown finite element method. The key steps of the classical stiffness method are:

1. Determine the DOFs of the structure, i.e., the unknown displacements and rotations, which are collected in the vector **u**
2. Establish the stiffness matrix, **K**, which contains the "stiffness coefficients" explained below
3. Establish the load vector, **F**, which contains the applied loads
4. Solve the system of equations **Ku**=**F** to obtain the unknown displacements and rotations
5. Draw the final section force diagrams

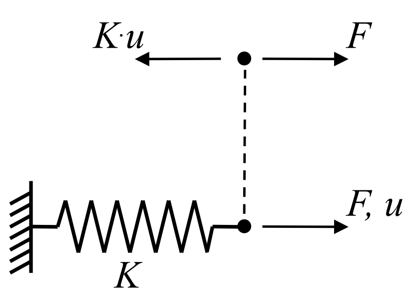

Figure 1: Simple spring with one DOF.

Concepts

As an introduction to the stiffness method, first consider a simple problem with only one DOF. In fact, consider the problem of a spring with stiffness K, as shown in Figure 1. Let F and u be the force and displacement along the DOF, respectively. As the spring is being pulled or compressed, a force equal to $K \cdot u$ develops in the spring. This force in the spring balances the applied force. Consequently, the equilibrium equation for this problem is

$$K \cdot u = F \tag{1}$$

Eq. (1), albeit in matrix form, is the central equation in the stiffness method. The following comments for the simple spring holds true in general. First, notice that the stiffness coefficient, K, represents the force due to a unit displacement. Second, observe that applied loads enter the right-hand side of Eq. (1) as positive when they act in the direction of the DOF. In other words, the applied load, F, in Figure 1 is positive because it acts in the direction of the DOF, u. However, the formal approach for establishing the loads in the equilibrium equations is slightly different. The DOFs of the structure are first *clamped*, i.e., the displacements and rotations are fixed. In turn, when applied loads act, there will be a clamping force. In these notes, the clamping forces are identified by a tilde symbol, as shown shortly. For the simple problem in Figure 1, because the clamping force restrains the structure against displacement u due to the force F, the clamping force is the negative of the force F in the right-hand side of Eq. (1):

$$F = -\tilde{F} \tag{2}$$

Consequently, the equilibrium equation is written equivalently in two ways:

$$\tilde{F} + K \cdot u = 0 \quad \Leftrightarrow \quad K \cdot u = F \tag{3}$$

Next, consider a 2-DOF problem, i.e., a problem with two DOFs, numbered 1 and 2. This means that two equilibrium equations are needed. In particular, for this problem the following two generic equilibrium equations are established by requiring equilibrium along the two DOFs:

$$\begin{aligned} \tilde{F}_1 + K_{11} \cdot \mathrm{u}_1 + K_{12} \cdot \mathrm{u}_2 = 0 \\ \tilde{F}_2 + K_{21} \cdot \mathrm{u}_1 + K_{22} \cdot \mathrm{u}_2 = 0 \end{aligned} \tag{4}$$

where $\tilde{F}_i$ is the clamping force along DOF number i due to external loads, K_{ij} is the force along DOF number i due to a unit displacement or rotation along DOF j, and u_j is the unknown displacement or rotation along DOF number i. This system of equations in Eq. (4) is written in matrix notation in the same two equivalent ways:

$$\tilde{\mathbf{F}} + \mathbf{K} \cdot \mathbf{u} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{K} \cdot \mathbf{u} = \mathbf{F} \tag{5}$$

where **K** is the stiffness matrix, **u** is the displacement vector, and **F** is the load vector. Again it is noted that the vector of clamping forces is the negative of the load vector.

Establishing the Stiffness Matrix

Once the DOFs of the structure are identified, in accordance with the document on degrees of indeterminacy, the stiffness matrix is established as follows:

1. Sketch the displaced shape of the structure for a unit displacement or rotation along DOF number j, with all other DOFs clamped
2. Determine the force along every DOF to maintain this displaced shape, i.e., K_{ij}, which form column number j of the stiffness matrix
3. Carry out Step 1 and 2 for all DOFs to establish all columns of the stiffness matrix
4. Check that the final stiffness matrix is symmetric and that it has positive components on the diagonal

The key challenge is Step 2, i.e., the determination of forces along the DOFs to maintain the displaced shape. For each DOF it is necessary to account for every force, including moments, from every member. Figure 2 provides stiffness-values for a few fundamental beam cases to assist this process. The stiffness values are derived from solving the differential equation, or equivalently by employing the slope-deflection equation, or the principle of virtual work. Each quantity in the auxiliary beam case is multiplied by the imposed displacement Δ or rotation θ to obtain the actual value of the force or moment. Axial stiffness is omitted from Figure 2 because in the classical stiffness method it is straightforward and often sufficiently accurate to neglect axial deformations in frame members. Axial deformations are often orders of magnitude less than the bending (flexural) deformations because the axial stiffness is large compared to the bending stiffness for many frame members. Axial deformations are neglected simply by considering the members infinitely stiff in the axial direction, so that the associated DOFs of the structure are removed. See details in the document on degrees of indeterminacy.

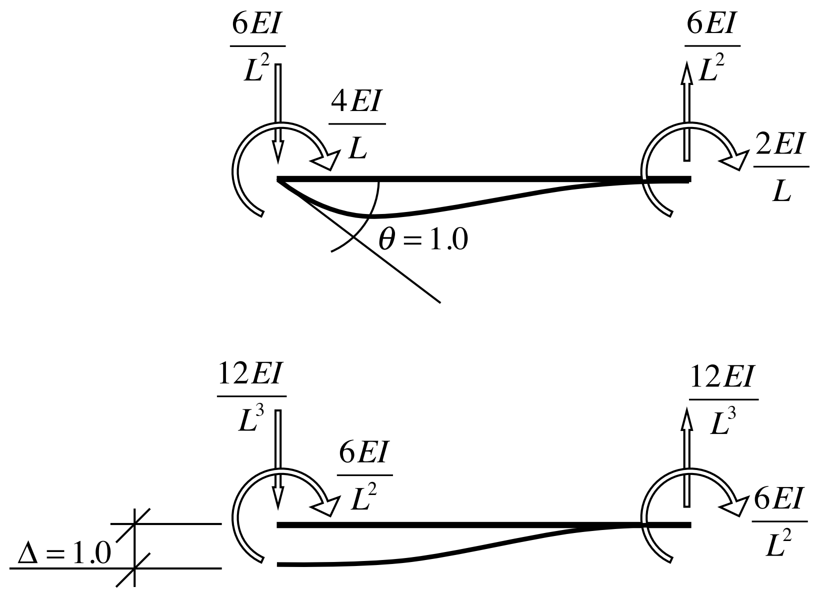

Figure 2: Stiffness values for fundamental beam cases.

Establishing the Load Vector

Two types of loads are considered when establishing the load vector: concentrated loads at the nodes (joints) and distributed member loads. The former are readily included by simply inserting them into the load vector $\mathbf{F}$, with a positive sign if they act in the direction of the DOF, otherwise negative. For distributed member loads the vector of clamping forces, $\tilde{\mathbf{F}}$, is first established. This is done as follows:

1. Clamp all the DOFs of the structure
2. Compute the force along each DOF to keep the structure clamped and insert them into $\tilde{\mathbf{F}}$
3. Flip the sign to obtain the load vector: $\mathbf{F} = -\tilde{\mathbf{F}}$

To assist the determination of clamping forces, an auxiliary document with fixed-end forces for several beam cases is provided elsewhere in these structural analysis notes. Other cases are derived by solving the differential equation for beam bending or utilizing the flexibility method.

Solution and Member Forces

Upon establishing the stiffness matrix, **K**, and the load vector, **F**, the linear system of equations is solved to obtain the displacement vector, **u**, which contains the displacements and rotations along the DOFs:

$$\mathbf{u} = \mathbf{K}^{-1} \cdot \mathbf{F} \tag{6}$$

Next, to determine the section forces in the members there are several approaches. One approach is to utilize the slope-deflection equation (see the document on the slope-deflection method):

$$M_{NF} = \frac{2EI}{L} \cdot \left(2\theta_N + \theta_F - 3 \cdot \psi\right) + FEM_{NF} \tag{7}$$

For each member, the applicable nodal displacements and rotations from **u** in Eq. (6) are inserted as θ_N, θ_F, and ψ in Eq. (7). Alternatively, in the computational stiffness method, the member forces are determined by invoking the local stiffness matrix for the member.