

# Size Effect Models

## Weibull Theory

This theory is based on a “weakest link” approach to material strength, and essentially simplifies the problem to the calibration of two parameters  $m$  and  $k$ . To understand the theory, consider a structural member that consists of  $n$  small elements. If any of the small elements fail, then the member fails. Denote the strength of each small element by  $\tau$  and assume that the probability distribution of this strength, i.e.,  $F(\tau)$  is known. Then, the survival probability of each element,  $p_{s,n}$  is

$$p_{s,n} = 1 - F(\tau) \quad (1)$$

If the small elements are statistically independent then the survival probability for the entire member is

$$p_s = (1 - F(\tau))^n \quad (2)$$

For large  $n$ , series expansion yields

$$p_s = e^{-n \cdot F(\tau)} \quad (3)$$

Consequently, the failure probability of the member, for large  $n$ , is

$$p_f = 1 - e^{-n \cdot F(\tau)} \quad (4)$$

According to Weibull, the CDF for the strength of each small element is

$$F(\tau) = \left( \frac{\tau}{m} \right)^k \quad (5)$$

for  $\tau > 0$ , where  $k > 0$  is called the shape parameter and  $m > 0$  is called the scale parameter. Eq. (5) is intended as a convenient approximation of the lower tail of the unknown actual probability distribution of the strength. Substitution of Eq. (5) into (4) yields

$$p_f = 1 - e^{-n \cdot \left( \frac{\tau}{m} \right)^k} \quad (6)$$

If the structural member is divided into infinitely many infinitesimally small elements then the failure probability reads:

$$p_f = 1 - e^{-\int_V \left( \frac{\tau}{m} \right)^k dV} \quad (7)$$

Under the assumption that the structural member has unit volume and uniform stress, the failure probability becomes:

$$p_f = 1 - e^{-\left( \frac{\tau^*}{m} \right)^k} \quad (8)$$

where  $\tau^*$  is the strength of the unit reference volume. Eq. (8) is the standard Weibull distribution. Suppose this reference strength is known. The strength of another structural member non-unit volume and/or non-uniform stress distribution is obtained by equating its failure probability with the unit-volume reference member:

$$e^{-\int_V \left(\frac{\tau}{m}\right)^k dV} = e^{-\left(\frac{\tau^*}{m}\right)^k} \Rightarrow \int_V \tau^k dV = (\tau^*)^k \quad (9)$$

Hence, a design check can be of the form

$$\tau^* \geq \left( \int_V \tau^k dV \right)^{\frac{1}{k}} \quad (10)$$

or equivalently of the form

$$\frac{\tau^*}{\left( \int_V \tau^k dV \right)^{\frac{1}{k}}} \geq 1.0 \quad (11)$$

To arrive at the O86 code equation for shear capacity of large glulam beams, multiply Eq. (11) by the constant  $W_f$  on both sides

$$W_f \cdot \frac{\tau^*}{\left( \int_V \tau^k dV \right)^{\frac{1}{k}}} \geq W_f \quad (12)$$

The shear stress distribution over a rectangular cross-section is

$$\tau(x, y, z) = \left( \frac{3}{2 \cdot A} - \frac{6 \cdot z^2}{A \cdot d^2} \right) \cdot V(x) \quad (13)$$

Hence, the volume integral in Eq. (12) reads

$$\begin{aligned} \int_V \tau^k dV &= b \cdot \int_{-\frac{d}{2}}^{\frac{d}{2}} \left( \frac{3}{2 \cdot A} - \frac{6 \cdot z^2}{A \cdot d^2} \right)^k dz \int_0^L V(x)^k dx \\ &\approx 3.3 \cdot \frac{A}{A^k} \cdot \int_0^L V(x)^k dx \\ &= 3.3 \cdot \frac{Z}{A^k} \cdot \frac{1}{L} \cdot \int_0^L V(x)^k dx \end{aligned} \quad (14)$$

Substitution into Eq. (12) yields

$$\tau^* \cdot A \cdot 3.3^{\frac{1}{k}} \cdot W_f \cdot \left( \frac{L}{\int_0^L V(x)^k dx} \right)^{\frac{1}{k}} \cdot Z^{\frac{1}{k}} \geq W_f \quad (15)$$

In contrast, the O86 code formula is

$$\begin{aligned} V_r &= \phi \cdot F_v \cdot 0.48 \cdot A_g \cdot K_N \cdot C_v \cdot Z^{-0.18} \\ &= 0.9 \cdot (f_v \cdot K_D \cdot K_H \cdot K_{Sv} \cdot K_T) \cdot 0.48 \cdot A_g \cdot (1.0) \cdot C_v \cdot Z^{-0.18} \\ &\geq W_f \end{aligned} \quad (16)$$

where the factor  $C_v$  is

$$C_v = 1.825 \cdot W_f \cdot \left( \frac{L}{\sum G} \right)^{0.2} = 1.825 \cdot W_f \cdot \left( \frac{L}{6 \cdot \int_0^L |V|^5 dx} \right)^{0.2} \quad (17)$$

where  $G$  is substituted by the integral of the shear force diagram because the code specifies

$$G = l_a \cdot (V_A^5 + V_B^5 + 4 \cdot V_C^5) \quad (18)$$

which is Simpson's integration rule without the factor 1/6. It is also noted that  $W_f$  is the total load,  $Z$  is the beam volume, and  $A_g$  is the cross-sectional area. In summary, the O86 code equation is

$$\phi \cdot F_v \cdot A \cdot 0.61 \cdot W_f \cdot \left( \frac{L}{\int_0^L |V|^5 dx} \right)^{0.2} \cdot Z^{-0.18} \geq W_f \quad (19)$$

Compared this with the theoretically derived Eq. (15).

### Possible Simpler Implementation

$$\int_V \tau^k dV = (\tau^*)^k \quad (20)$$

$$\tau^* \geq \left( \int_V \tau^k dV \right)^{\frac{1}{k}} = \left( 3.3 \cdot \frac{A}{A^k} \cdot \int_0^L V(x)^k dx \right)^{\frac{1}{k}} \quad (21)$$

$$\tau^* \geq 1.27 \cdot A^{\frac{1-k}{k}} \cdot \left( \int_0^L V(x)^k dx \right)^{\frac{1}{k}} \quad (22)$$

### Determination of Shape and Scale Parameters

The shape parameter,  $k$ , and the scale parameter,  $m$ , are determined from tested beams with known volume, shear force diagram, and shear failure load. To calculate  $k$  and  $m$  the stress at a reference location in the beam due to the failure load is first computed. The maximum shear stress in the cross-section is typically selected, which for rectangular cross-sections read

$$\tau_M = \frac{3V}{2A} \quad (23)$$

Next, this reference stress enters a generic equation that expresses the stress at any location in the beam:

$$\tau(x, y, z) = \tau_M \cdot \theta(x, y, z) \quad (24)$$

Substitution into Eq. (9) yields

$$\int_V (\tau_M \cdot \theta)^k dV = (\tau^*)^k \quad \Rightarrow \quad \tau_M = \frac{\tau^*}{\left( \int_V \theta^k dV \right)^{1/k}} \quad (25)$$

Factorize out the shear force from the integral and introduce  $I(k)$ , and equivalently  $\beta$ :

$$\tau_M = \frac{\tau^*}{(I(k) \cdot V)^{1/k}} = \beta(k) \cdot \frac{\tau^*}{(V)^{1/k}} \quad (26)$$

In a plot with  $\ln(\tau_M)$  along the abscissa axis versus  $\ln(V)$  along the ordinate axis the slope of the function is  $1/k$ . To ease the regression, the median  $\tau_M$  for each beam configuration is usually plotted. Knowing  $k$ ,  $\beta$ , and the median values of  $\tau_M$ , the corresponding media values of  $\tau^*$  are computed from Eq. (25). This gives the median value of the Weibull-distributed random variable  $\tau^*$ , which corresponds to the inverse CDF value at 0.5 probability of exceedance

$$0.5 = 1 - e^{-\left(\frac{\tau_{\text{median}}^*}{m}\right)^k} \quad (27)$$

Solving for  $m$  yields:

$$m = \frac{\tau_{\text{median}}^*}{(-\ln(1 - 0.5))^{1/k}} \quad (28)$$

Regression results from Foschi and Barrett for  $1.0\text{m}^3$  reference volume is  $k=5.53$  and  $m=2,540\text{kN/m}^2$ . The values for  $k$  and  $m$  are substituted into the Weibull equation, i.e., Eq. (8), to obtain reference strength values for different exceedance probabilities:

$$\tau^*(p) = m(-\ln(1 - p))^{1/k} \quad (29)$$

For example:

$$\begin{aligned}\tau_{0.5}^* &= (2,540\text{kN/m}^2)(-\ln(1 - 0.5))^{1/5.53} = 2,377\text{kN/m}^2 \\ \tau_{0.05}^* &= (2,540\text{kN/m}^2)(-\ln(1 - 0.05))^{1/5.53} = 1,484\text{kN/m}^2\end{aligned}\tag{30}$$

## References

Weibull

Foschi and Barrett