Set Theory

Definitions

Set theory is a language with which we can talk about events. In this sense, it underpins the theory of probability, where the likelihood of events is in question.

- The set of all possible events is called the sample space, S
- Each individual sample point is denoted x
- Any collection of sample points is called an event, typically denoted E, perhaps with a subscript to further identify the event
- The complement of an event, denoted $\bar{E}$, is an event that contains all the sample points that are not in E.
- The event S is called the certain event because it is bound to occur
- The event ø is the null event; it contains no sample points and has no possibility of occurring.

There are two types of sample spaces:

- Discrete (finite or infinite)
- Continuous

Venn Diagrams

Venn diagrams are useful for visualizing events, as shown in Figure 1.

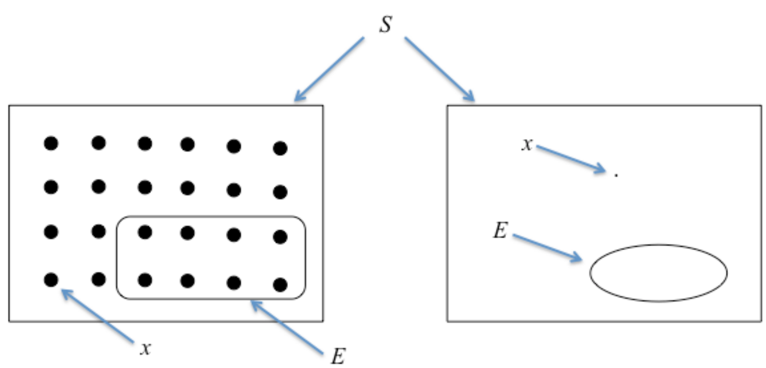

Figure 1: Venn diagram for discrete (left) and continuous (right) sample spaces.

Operations

The union and intersection operations create compound events out of two or more events. The union of two events, denoted $E_1 \cup E_2$, is an event that contains all the sample points that are either in E_1 or in E_2. It is therefore pronounced "or." The intersection of two events, denoted $E_1 \cap E_2$ or E_1E_2 in shorthand notation is an event that contains all the sample points that are both in E_1 and in E_2. It is therefore pronounced "and." For a continuous sample space the union and intersection operations are grey-shaded in the Venn diagrams in Figure 2.

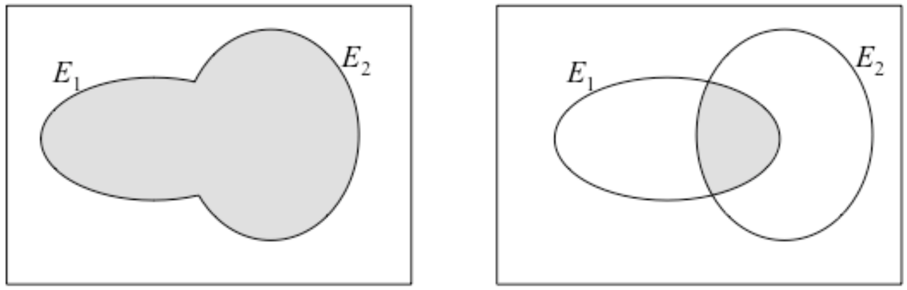

Figure 2: Visualization of union and intersection events.

The union and intersection operators obey the following rules:

- Commutative rule:

$$E_1 \cup E_2 = E_2 \cup E_1 \tag{1}$$

$$E_1 \cap E_2 = E_2 \cap E_1 \tag{2}$$

- Associative rule:

$$\left(E_1 \cup E_2\right) \cup E_3 = E_1 \cup \left(E_2 \cup E_3\right) \tag{3}$$

$$\left(E_1 \cap E_2\right) \cap E_3 = E_1 \cap \left(E_2 \cap E_3\right) \tag{4}$$

- Distributive rule:

$$\left(E_1 \cup E_2\right) \cap E_3 = \left(E_1 \cap E_3\right) \cup \left(E_2 \cap E_3\right) \tag{5}$$

$$\left(E_1 \cap E_2\right) \cup E_3 = \left(E_1 \cup E_3\right) \cap \left(E_2 \cup E_3\right) \tag{6}$$

To avoid excessive use of parentheses, in the same way as multiplication takes precedence over addition, intersection takes precedence over union operations.

MECE

The terms "mutually exclusive" and "collectively exhaustive" are often used:

- Two events are said to be mutually exclusive if their intersection is the null event
- Two events are collectively exhaustive if their union constitutes the entire sample space

De Morgan's Rule(s)

The following rules allow us to transform union of events into intersection of events, and vice versa. The fundamental version of de Morgan's rule states: "the complement of a union is equal to the intersection of the complements:"

$$\overline{E_1 \cup E_2} = \overline{E}_1 \cap \overline{E}_2 \tag{7}$$

This rule is verified by visualizing both sides of Eq. (7) in a Venn diagram, where it is observed that they both refer to the same event. The rule is generalized to

$$\overline{E_1 \cup E_2 \cup \cdots \cup E_n} = \overline{E}_1 \overline{E}_2 \cdots \overline{E}_n \tag{8}$$

by repeated application of Eq. (7). Another version of de Morgan's rule, i.e., another rule, states: "the complement of an intersection is equal to the union of the complements:

$$\overline{E_1 \cap E_2} = \overline{E}_1 \cup \overline{E}_2 \tag{9}$$

This rule is derived by applying Eq. (7) to two complements:

$$\overline{\overline{E}_1 \cup \overline{E}_2} = \overline{\overline{E}}_1 \cap \overline{\overline{E}}_2 = E_1 \cap E_2 \tag{10}$$

Taking the complement on both sides yield Eq. (9).