Reliability Analysis

In the context of these notes, the word reliability originates in the structural safety community, where the primary objective is to compute the failure probability, denoted p_f. The reliability is the complement of the failure probability:

$$\text{Reliability} = 1 - p_f \tag{1}$$

The reliability problem, i.e., the problem of computing p_f, has two ingredients: random variables that describe the uncertainty and one or more limit-state functions that define failure. If the problem has only one limit-state function then the problem is referred to as a component reliability problem. Otherwise it is a system reliability problem. It is important to note that in modern reliability analysis the failure event is not necessarily structural collapse or some other event that is easily associated with the word failure. Rather, limit-state functions are defined to compute the probability of a range of events, such as the monetary loss exceeding some selected threshold.

Limit-state Functions

Since the early 80's, when "allowable stress design" was replaced by "limit-state design," structural engineers have become familiar with many "limit-states." Limit-states related to strength are often called ultimate limit-states (ULS), while those related to deflections and vibrations are called serviceability limit-states (SLS). More generally, a limit-state function defines what the engineer considers to be failure. In other words, in reliability analysis the limit-state function defines the event for which the probability is sought. Denoting the limit-state function by $g(\mathbf{x})$, where $\mathbf{x}$ is the vector of realizations of the random variables, the syntax is:

$$\begin{aligned} g(\mathbf{x}) < 0 &: \text{ Failure} \\ g(\mathbf{x}) > 0 &: \text{ Safe} \\ g(\mathbf{x}) = 0 &: \text{ The limit-state surface} \end{aligned} \tag{2}$$

One example of a limit-state function is

$$g = u_0 - u(\mathbf{x}) \tag{3}$$

where u_0 is a threshold value and u is a response value that depends on the outcome of the random variables. A reliability analysis with this limit-state function yields and estimate of the probability that u exceeds the threshold u_0.

The same reliability problem can be posed by different but equivalent limit-state functions, which all satisfy Eq. (2). One example is the basic reliability problem; equivalent limit-state functions for this problem include

$$g = R - S \quad \Leftrightarrow \quad g = 1 - \frac{S}{R} \quad \Leftrightarrow \quad g = \ln\left(\frac{R}{S}\right) \tag{4}$$

All these limit-state functions are negative when S exceeds R, otherwise they are positive. Therefore, because they all correctly identify the failure event, they are called "equivalent limit-state functions."

The Basic Reliability Problem

A simple but fundamental problem in reliability analysis is the "basic reliability problem," which is defined by the limit-state function

$$g = R - S \tag{5}$$

where R, from the French word *résistance*, is a random variable that represents resistance, i.e., capacity, and S, from the French word *sollicitation*, is a random variable that represents load, i.e., demand. It is observed that g is negative when S exceeds R, hence this limit-state function appropriately defines the failure event. Under the assumption that the resistance and load are statistically independent, in which case the joint PDF is $f(r,s)=f(r) \cdot f(s)$, the problem specializes to

$$p_f = \int_{R<S}\int f_R(r) f_S(s)\, dr\, ds = \int_0^\infty \int_0^s f_R(r) f_S(s)\, dr\, ds = \int_0^\infty F_R(s) f_S(s)\, ds \tag{6}$$

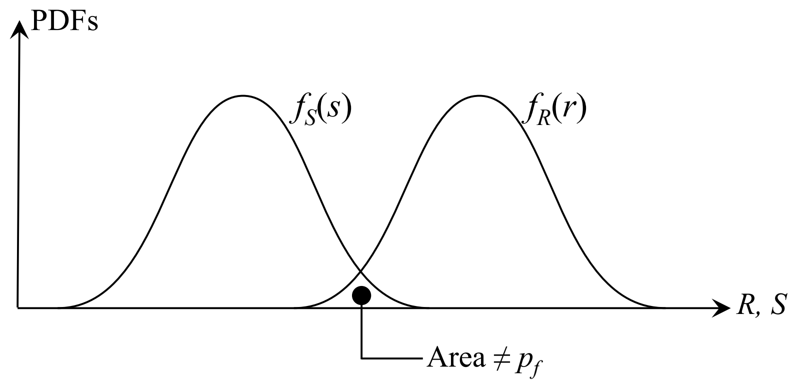


Figure 1: Visualization of the basic reliability problem by individual PDFs.

Eq. (6) shows that the failure probability for the basic reliability problem is not the overlapping area between the two marginal PDFs shown in Figure 1. Nor is it the overlapping area between the resistance-PDF and the load-CDF. Rather, the failure probability is the solution to an integral, i.e., the one in Eq. (6). To visualize this integral, it is more instructive to view the plane stretched by the two random variables, as shown in Figure 2.

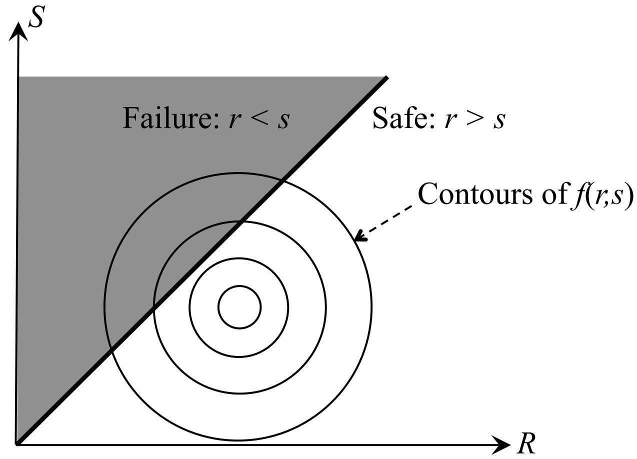


Figure 2: Visualization of the basic reliability problem in the random variable space.

Now consider the situation where R and S have the Normal probability distribution. In this case, g is also Normal because it is a linear function of R and S. In fact, the mean of g is:

$$\mu_g = \mu_R - \mu_S \tag{7}$$

and the variance of g is:

$$\sigma_g^2 = \nabla g^T \mathbf{\Sigma}_{RS} \nabla g = \sigma_R^2 + \sigma_S^2 \tag{8}$$

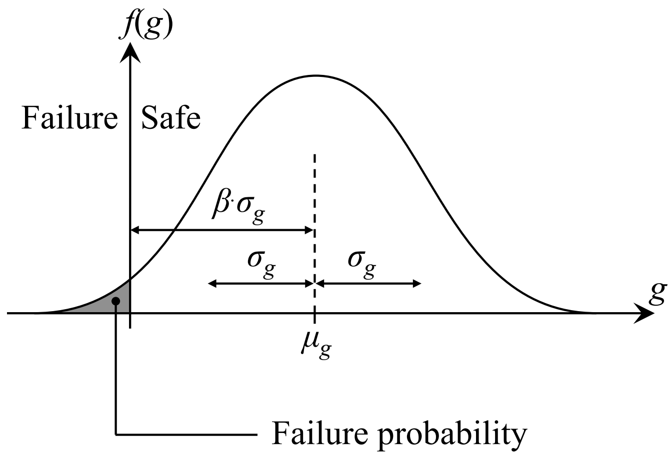
This implies that the failure probability is computed from the standard normal CDF, Φ:

$$p_f = \mathrm{P}(g \leq 0) = \Phi\left(\frac{g-\mu_g}{\sigma_g}\right) = \Phi\left(-\frac{\mu_g}{\sigma_g}\right) = \Phi(-\beta) \tag{9}$$

where the all-important reliability index, β, is defined as

$$\beta \equiv \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{10}$$

This reliability index, in the context of the basic reliability problem, is interpreted in Figure 3. It is observed that β is the number of standard deviations from the failure domain (grey shaded) to the mean. Clearly, the greater this distance is the greater the safety is. In other words, a high reliability index implies a low failure probability. This relationship is visualized in Figure 4, where β is plotted against p_f in the range of typical β-values. The exact relationship in Eq. (9) is shown by a solid line, and the gross approximation p_f=$10^{-\beta}$ is shown as a dashed line. The approximation is not used in practice because it is observed in Figure 4 that it is only a valid estimate around β=3.



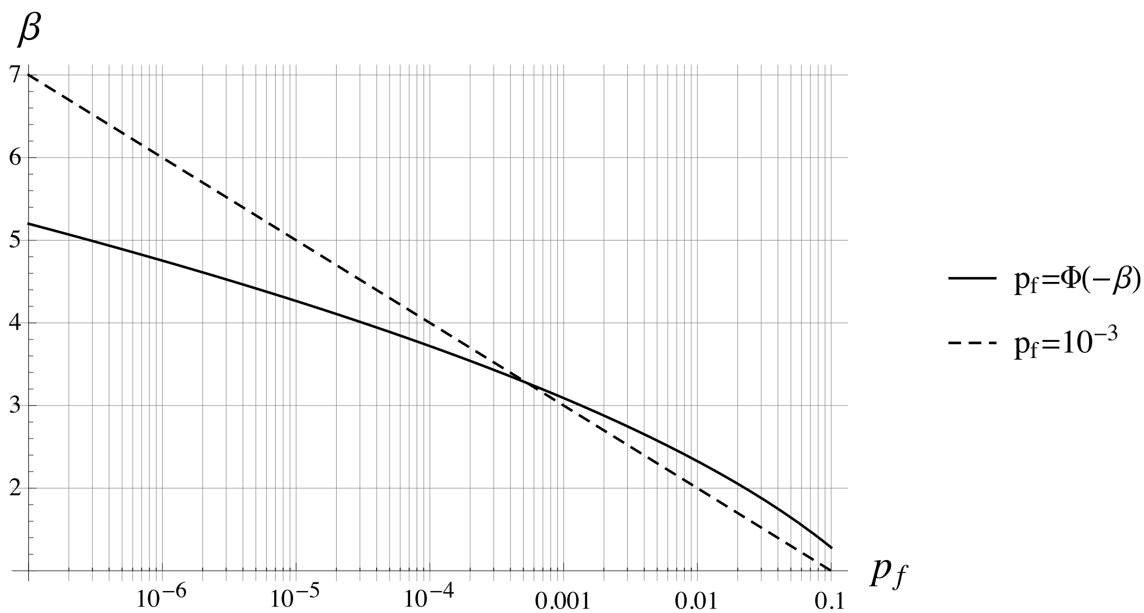


Figure 3: Normal probability distribution for the basic limit-state function.

Figure 4: Relationship between β and p_f.

The General Reliability Problem

A generic reliability problem with only one limit-state function is referred to as the "component reliability problem," which reads:

$$p_f = \mathrm{P}(g \leq 0) = \int_{g\leq 0} \cdots \int f(\mathbf{x})\, d\mathbf{x} \tag{11}$$

where $f(\mathbf{x})$ is the joint PDF for the random variables. Problems in which the joint state of more than one limit-state function defines failure are called "system reliability problems," and they are addressed in another document. Eq. (11) shows that the component reliability problem amounts to integrating the probability density in the region of the **x**-space where the limit-state function is negative. All reliability methods address this problem. Unfortunately, it is impossible to solve the multi-fold integral in Eq. (11) analytically, except for a few special cases. However, methods like FORM, SORM, and sampling provides approximate solutions.

Reliability Methods

Because it is possible to solve Eq. (11) analytically for only a few special cases, a host of reliability methods have been developed to solve the component reliability problem in an approximate manner. Some of these are listed below and described in other documents.

- The mean-value, first-order, second-moment (MVFOSM) method employs only second-moment information about the random variables together with a first-order approximation of the limit-state function about the mean of the random variables. This method is highly efficient, but it is associated with several disadvantages: 1) It may lead to different results for equivalent limit-state functions, referred to as the invariance problem; 2) The linearization of the limit-state function may lead to inaccuracies when the limit-state function is nonlinear; 3) Information about the distribution type of the random variables is ignored.
- The first-order reliability method (FORM) remedies the invariance problem of MVFOSM and incorporates information about the distribution type of the random variables. These advantages come at a higher computational cost, accompanied by the possibility of non-convergence. FORM maintains the linearization of the limit-state function and may thus lead to inaccuracies when the limit-state function is nonlinear.
- The second-order reliability method (SORM) is equivalent to FORM except that the limit-state function is approximated by a second-order function. To some extent this remedies possible inaccuracies when the limit-state function is nonlinear, but more computations are necessary compared with FORM.
- Sampling methods evaluate the limit-state function at many realizations of the random variables and yield an approximate value of the failure probability. Several sampling methods are available, ranging from mean-centered Monte Carlo sampling to adaptive sampling schemes. A particularly appealing sampling method when FORM as been carried out is importance sampling. Based on information from the FORM analysis this approach provides an accurate and robust estimate of the failure probability with far fewer samples than, say, Monte Carlo sampling.
- Response surface methods, including the neural network approach, are three-step approaches. First, the limit-state function is evaluated at many pre-selected realizations of the random variables. Next, some interpolation function, typically a second-order function or a neural network, is utilized to approximate the limit-state function. Finally, a reliability method, like FORM or sampling, is employed in conjunction with the approximate function to estimate the failure probability.

Although it is somewhat artificial in practical use, several textbooks distinguish between Level I, II, III, and IV reliability methods:

- Level I methods: In these methods only one value associated with each uncertain parameter enters the analysis. For example, design code equations that utilize one characteristic value for each parameter represent Level I methods. Level I methods do not yield a failure probability.

- Level II methods: Two values for each uncertain parameter enter. Typically, the mean and standard deviation for each parameter, in addition to possible correlation between the parameters. According to this definition, second-moment methods are Level II methods. Unless the random variables are normally distributed, Level II methods do not yield a failure probability.
- Level III methods: At this level the joint probability distribution for the parameters are available and a failure probability is produced.
- Level IV methods: These go beyond Level III methods by including cost-benefit considerations that yield the target reliability. This approach is suitable for projects with limited precedence or dramatic failure consequences, such as offshore oil and gas installations and nuclear power plants.