Quadrature

Quadrature is another name for numerical integration. Some quadrature rules are generally useful for computing cumbersome definite integrals. However, some quadrature rules play a particularly important role in the finite element method, which is a popular and advanced method of structural analysis. In the following, the following three integrals are considered

$$\int_{-1}^{1} f(x)\,dx \tag{1}$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy \tag{2}$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\,dx\,dy\,dz \tag{3}$$

The first is single-fold, the second is two-fold, and the third is a three-fold integral. It is also noted that the integral boundaries are –1 and 1. Clearly, not all integrals have these boundaries. To transform the integral of a function $f(\tilde{x})$ with boundaries a and b into and integral along the x from –1 to 1 the following axis transformation is easily established:

$$\tilde{x} = \left(\frac{b-a}{2}\right)\cdot x + \left(\frac{b+a}{2}\right) \tag{4}$$

In addition to substituting Eq. (4) into the function $f(\tilde{x})$ it is necessary to transform the integral differentials. In particular, the integrand is multiplied by the determinant of the Jacobian matrix, namely $J=|\mathbf{J}|$. For the three integrals in Eqs. (1) to (3) it reads, respectively:

$$J = \frac{d\tilde{x}}{dx} = \left(\frac{b-a}{2}\right) \tag{5}$$

$$J = \begin{vmatrix} \left(\frac{b_1-a_1}{2}\right) & 0 \\ 0 & \left(\frac{b_2-a_2}{2}\right) \end{vmatrix} = \left(\frac{b_1-a_1}{2}\right)\cdot\left(\frac{b_2-a_2}{2}\right) \tag{6}$$

$$J = \begin{vmatrix} \left(\frac{b_1 - a_1}{2}\right) & 0 & 0 \\ 0 & \left(\frac{b_2 - a_2}{2}\right) & 0 \\ 0 & 0 & \left(\frac{b_3 - a_3}{2}\right) \end{vmatrix} = \left(\frac{b_1 - a_1}{2}\right) \cdot \left(\frac{b_2 - a_2}{2}\right) \cdot \left(\frac{b_3 - a_3}{2}\right) \quad (7)$$

where subscripts are introduced to distinguish the integration boundaries in different integration directions.

Trapezoidal Rule

(Not written yet.)

Simpson's Rule

(Not written yet.)

Gauss Quadrature

Gauss quadrature is the most popular scheme for integration of 2D and 3D finite elements. Consider first Gauss quadrature for the single-fold integral in Eq. (1). The integration rule is written

$$\int_{-1}^{1} f(x)\ dx = \sum_{i=1}^{N} w_i \cdot f(x_i) \quad (8)$$

where N is the number of integration points, w_i are integration weights, and x_i are integration points.

Table 1 provides the location of the integration points and associated integration weights up to N=4. Two and three-fold integrals are evaluated by applying the same points and weights in orthogonal directions:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j f(x_i, y_j) \quad (9)$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\,dx\,dy\,dz = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} w_i w_j w_k f(x_i, y_j, z_k) \quad (10)$$

Table 1: Integration points and weights for Gauss quadrature.

N	x_i	w_i
1	0	2
2	–0.577350269189626	1

	+0.577350269189626	1
3	–0.774596669241483 0 +0.774596669241483	0.555555555555556 0.888888888888889 0.555555555555556
4	–0.861136311594053 –0.339981043584856 +0.339981043584856 +0.861136311594053	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454

Gauss integration provides exact results for integration of polynomials up to order $2n$-1, where n is the number of integration points in one direction.

Gauss-Lobatto Quadrature

(Not written yet.)