Notation

In general, symbols that represent a scalar physical quantity, such as a length, L, are written in Italics. Functions and operators, such as the probability, P( ), and the sine function, sin( ), are not. When vector and matrix notation is employed then vectors and matrices are written in boldface, such as the matrix **D**. As explained later in this document, index notation employs Italics, like with symbols, e.g., D_{ij}.

Vector and Index Notation

An important class of objects in engineering analysis is vectors and matrices. More generically they are referred to as tensors. These objects are expressed either in vector notation or in index notation. Sometimes the phrase matrix notation is used instead of vector notation. Similarly, tensor notation is sometimes used instead of the phrase index notation.

Vector notation is most appealing to the untrained eye. Boldface clearly identifies vectors and matrices. One example is **Ax**=**b**. In vector notation, upper case usually refers to matrices and lower case refers to vectors. However, although it is visually appealing vector notation is somewhat limiting in rigorous analysis. In such circumstances index notation is an appealing alternative.

In index notation, a vector **a** is written a_i, where i is an index that runs through non-negative integers. Depending on convention, the index starts at 0 or 1. In index notation, a matrix **A** is written A_{ij}, where the first index is the row number and the second index is the column number. Tensors of higher dimensions follow naturally, such as C_{ijk}.

A key advantage of index notation is that each object—that being a vector, matrix, or higher-order tensor—is treated as a scalar. After the calculations are carried out, the result can be presented in vector notation for visual appeal. Index notation is also appealing when the equations are implemented on the computer. Structured programming blocks, such as for-loops, can use the same indices as those in the index notation.

Einstein's summation convention is vital to the attractiveness of index notation. To understand this rule, consider first a vector-matrix product. Like in ordinary algebra, two vectors or matrices written adjacent to each other implies multiplication. For example, **Ax=b** means that the matrix **A** should be multiplied by the vector **x** to get the vector **b**. Specifically, the first component of **b** is the sum of the products of the components of the first row of **A** with the components of **x**. By hand, this product is conveniently carried out in a Gant diagram:

$$\begin{array}{ccc|c} & & & x_1 \\ & & & x_2 \\ & & & x_3 \\ \hline A_{11} & A_{12} & A_{13} & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21} & A_{22} & A_{23} & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ A_{31} & A_{32} & A_{33} & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{array} \tag{1}$$

The Gant diagram becomes particularly attractive when more than two vectors/matrices are multiplied by hand calculation. By using the summation symbol, the product is written

$$b_i = \sum_{j=1}^{3} A_{ij} a_j \tag{2}$$

Einstein's summation convention allows Eq. (2) to be written in shorthand as

$$b_i = A_{ij} a_j \tag{3}$$

because the rule states that summation should be taken over any index that is repeated in a term. In this case that is *j*, because it appears twice in the term. This makes *j* a dummy index in Eq. (3); it serves only the summation purpose. Conversely, *i* is called a free index because it appears only once in each term. In the following examples of multiplication in vector and index notation, notice which are the free indices, and which are dummy indices. Also notice the following:

- The free indices reveals the dimension of the final result
- Unless there is a transpose the neighbouring indices are the same
- Transpose switches order of the indices of a matrix
- If there is a transpose, then neighbouring indices are no longer identical, but the range of the neighbouring indices must be the same
- The transpose of a vector does not affect the index notation; is an artifact of vectors being considered column vectors by default
- The ordering of the objects, i.e., vectors and matrices, in index notation is immaterial, while the order of the objects in vector notation is crucial

Free: *ik*
$$\mathbf{AB} = A_{ij} B_{jk} \tag{4}$$

Free: *im*
$$\mathbf{ABC} = A_{ij} B_{jk} C_{km} \tag{5}$$

Free: *i*
$$\mathbf{Ax} = A_{ij} x_j \tag{6}$$

Free: *ik*
$$\mathbf{A}^T \mathbf{B} = A_{ji} B_{jk} \tag{7}$$

Free: *i*
$$\mathbf{A}^T \mathbf{x} = A_{ji} x_j \tag{8}$$

Free: ik $$(\mathbf{AB})^T = \left(A_{ij}B_{jk}\right)^T = A_{kj}B_{ji} = B_{ji}A_{kj} = \mathbf{B}^T\mathbf{A}^T \tag{9}$$

Free: none (scalar) $$\mathbf{x}^T\mathbf{A}\mathbf{x} = x_i A_{ij} x_j \tag{10}$$

Free: none (scalar) $$(\mathbf{Ax})^T(\mathbf{Ax}) = A_{ij}x_j A_{ik}x_k = x_j A_{ji} A_{ik} x_k = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} \tag{11}$$

Notice in particular the steps taken in Eq. (9). In the first equality it is decided that i and k are the free indices, hence the final result must have the indices ik, in that order. In the second equality the indices ik are flipped to ki because of the transpose. In the third equality, the order of B and A is flipped to achieve a final result that has indices ik. In the last equality it is recognized that both A and B must be transposed to make the dummy index j in A be a neighbour to the dummy index j in B.

Notation for Derivatives

The ordinary way of writing a derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{or} \quad \frac{\partial}{\partial x} \tag{12}$$

for ordinary and partial differentiation, respectively. For derivatives with respect to time, x is replaced with t. However, notational shortcuts are often employed. For spatial derivatives, the most common ones apply the prime and the comma. Each prime denotes one differentiation with respect to the obvious variable, for example:

$$f'(x) \quad \text{and} \quad f''(x) \tag{13}$$

In contrast, the use of comma is more closely associated with index notation, and shows explicitly the variable with respect to which differentiation is carried out. For example,

$$\sigma_{ij,i} \tag{14}$$

states that the stress component σ_{ij} is differentiated in the i-direction. Such derivatives with respect to a vector of variables, but of a single function of those variables, say $h(\mathbf{x})$, are sometimes written by the nabla symbol:

$$\frac{\partial h}{\partial x_i} = \nabla_{\mathbf{x}} h \tag{15}$$

Conversely, the common short-hand notation for derivative with respect to time is a superimposed dot:

$$\begin{aligned} &\text{Displacement:} && u \\ &\text{Velocity:} && \dot{u} \\ &\text{Acceleration:} && \ddot{u} \end{aligned} \tag{16}$$

Mathematical Notation

Let $x \in \mathbb{R}_+^4$, where $\mathbb{R}_+^4 = \{x \in \mathbb{R}^n \mid x_i \geq 0, \forall i\}$, and let $q, w \in \mathbb{R}$. Furthermore, let $f : x, q \mapsto w$. Consequently, $f : \mathbb{R}_+^4 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This is one way of stating that a quantity, w, is determined by some function, f, which takes as input four positive numbers that are collected in a vector, **x**, as well as a quantity q. This is a mathematical way of describing how a displacement, w, is obtained from a load, q, acting on a beam with four properties collected in the vector **x**, presumably cross-sectional dimensions, member length, and modulus of elasticity. Such mathematical language is useful in rigorous and compact engineering analysis. However, in the beginning in can make the material harder to understand. Do not despair; the prominent mathematician John von Neumann said that: *"In mathematics you don't understand things. You just get used to them."* This section is intended to provide help in that process. The most commonly used symbols are:

$=$	"is equal to"
$\neq$	"is not equal to"
$\approx$	"is approximately equal to"
$\propto$	"is proportional to"
$\equiv$	"is defined as" (this can also be written $\triangleq$)
$<$	"is less than"
$>$	"is greater than"
$\ll$	"is much less than"
$\gg$	"is much greater than"
$+$	"plus"
$-$	"minus"
$\cdot$	"multiplied by"
$/$	"divided by"
$\mid$	"given that"
$\times$	the cross product of vectors or the Cartesian product
$:$	"is such that"
$\forall$	"for all"
$\mapsto$	"maps the element … into the element …"
$\to$	"maps the set … into the set …"
$\mathbb{R}$	the set of real numbers ($\mathbb{R}^n$ is an n-dimensional space of real numbers)
$\mathbb{Z}$	the set of integers ($\mathbb{Z}^n$ is an n-dimensional space of integers)
$\mathbb{C}$	the set of complex numbers ($\mathbb{C}^n$ is an n-dimensional space of complex numbers)
$\mathbb{Q}$	the set of rational numbers, i.e., numbers that can be expressed as a fraction ($\mathbb{Q}^n$ is an n-dimensional space of rational numbers)
$\mathbb{N}$	the set of natural number, i.e., non-negative integers ($\mathbb{N}^n$ is an n-dimensional space of natural numbers)
∂	partial derivative
∇	"nabla" or "the gradient of" that returns the vector of partial derivatives with respect to the function's parameters
$\nabla_{\mathbf{x}}$	same as the nabla operator, but specifically identifying the vector of parameters that differentiation is with respect to

$\exists$	"there exists"
$\in$	"is an element of"
$\subset$	"is a subset of"
$\bigcup$	"union"
$\bigcap$	"intersection"
$\varnothing$	the null set
$\sum$	"sum of"
$\prod$	"product of"
∞	"infinity"
$\lVert \; \rVert$	"the norm of" (for vectors, the square root of sum of squares)