Nonlinear Equations

While linear equations—and systems thereof—are covered by linear algebra, the problem of finding roots of nonlinear equations is addressed here. The word "root" denotes the value(s) of the unknown(s) that make the left-hand side of the equation(s) equal to the right-hand side. Depending on the problem there is one solution, many solutions, no unique solution, or no solution at all.

Quadratic Equations

Consider a generic second-order equation with one unknown, x:

$$a \cdot x^2 + b \cdot x + c = 0 \tag{1}$$

The two roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \tag{2}$$

Nonlinear Equations: Newton's Method

Consider a generic nonlinear equation of the form $f(x)=0$. Newton's iterative algorithm to find the root(s) is based on a linear Taylor approximation of the function about the point x_n:

$$f(x) \approx f(x_n) + f'(x_n) \cdot (x - x_n) \tag{3}$$

Setting this approximation equal to zero, which is the objective of the analysis, and solving for x yields

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

If $f(x)$ is a linear function then Eq. (4) provides the root x without iterations. Otherwise, the iterative Newton's algorithm is deviced:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{5}$$

which is based on successive linear linearizations at point x_1, x_2, … Convergence is approved when the difference between x_{n+1} and x_n is sufficiently small.