

# Moment-Area Method

This method computes bending deformation in frame members with known bending moment diagram. In fact, for any structure with known bending moment diagram this is an appealing method for determining displacements and rotations. For example, given the bending moment diagram from a computational structural analysis for a complex frame the moment-area method is ideal for extracting information about the deflections. The alternative virtual work method is often less convenient in such situations because it requires a new analysis of the structure with a unit load applied where the deformation is sought. Adding to the positive is that the moment-area method helps develop intuition and understanding of the deformation of structures. On the downside, the moment-area method lacks a step-by-step procedure; unique considerations are necessary for each structure.

## Moment-Area Theorems

The moment-area method is based on two theorems; the first is used to find rotations and the second is used to find displacements. To derive the first theorem, consider the differential equation for beam bending without equilibrium equations:

$$\frac{d^2 w}{dx^2} = \frac{M}{EI} \quad (1)$$

Furthermore, from Euler-Bernoulli beam theory the rotation of the cross-section of a cantilevered infinitesimal beam element equals the slope of the neutral axis:

$$\frac{dw}{dx} = \tan(\theta) \approx \theta \quad (2)$$

Combining Eqs. (1) and (2) yields the relationship between bending moment and rotation:

$$\frac{d\theta}{dx} = \frac{M}{EI} \quad (3)$$

Rearranging, the infinitesimal change in rotation for every infinitesimal element along the beam is obtained:

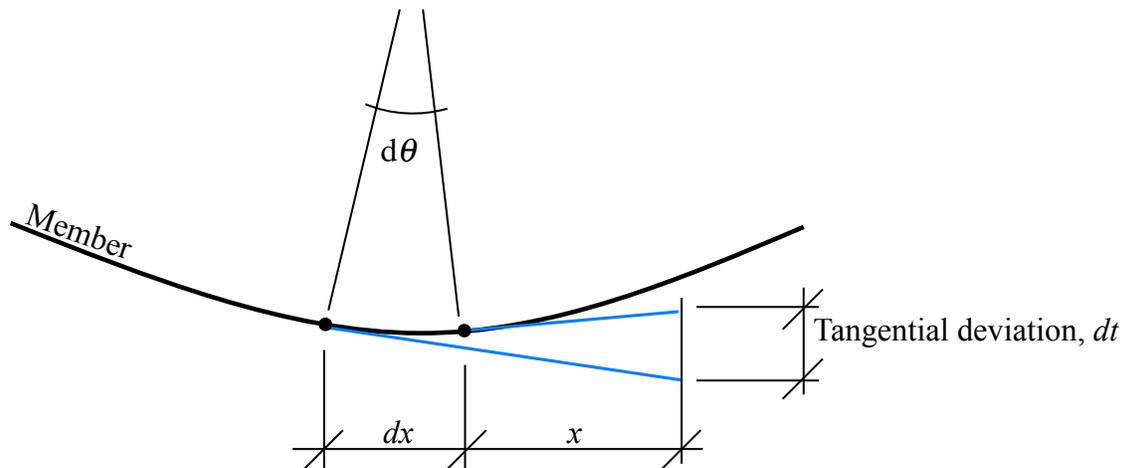
$$d\theta = \frac{M}{EI} \cdot dx \quad (4)$$

Integrating along the member yields the change in rotation between two points  $A$  and  $B$ :

$$\theta_{AB} = \int_A^B d\theta = \int_A^B \frac{M}{EI} \cdot dx \quad (5)$$

**This is the first moment-area theorem:** The change in rotation between two points  $A$  and  $B$  equals the area of the  $M/EI$  diagram between  $A$  and  $B$ . A caveat applies to members with hinges; Eq. (5) is invalid if a hinge is located between  $A$  and  $B$ .

The second theorem of the moment-area method introduces the concept “tangential deviations.”



**Figure 1: Tangential deviation.**

Essentially, a tangential deviation is the distance between two tangent lines. In Figure 1, the tangential deviation is the distance between the tangents coming from two points that are spaced at an infinitesimal distance. Clearly, the size of the tangential deviation depends upon how far from the points it is measured. In Figure 1 it is measured at a distance  $x$  from the points. Since the two points are infinitesimally close together, the two tangent lines can be considered to originate at one location. Consequently, the arc length at distance  $x$ —when  $x$  is interpreted as a radius—is  $(d\theta)(x)$ . Assuming small deformations, so that the geometrical considerations are made in the beam’s un-deformed configuration, the vertical distance between the two lines, i.e., the tangential deviation, is equal to the arc length:

$$dt \approx d\theta \cdot x \quad (6)$$

Substitution of Eq. (4) yields

$$dt = \frac{M}{EI} \cdot x \cdot dx \quad (7)$$

By integrating tangential deviations between two points  $A$  and  $B$  one obtains

$$t_{BA} = \int_A^B dt = \int_A^B \frac{M}{EI} x \, dx \quad (8)$$

where  $x$  is defined in Figure 1.

**This is the second moment-area theorem:** The tangential deviation at some location  $C$  between a tangent line from  $B$  and a tangent line from  $A$  is equal to the moment about  $C$  of the  $M/EI$  diagram between  $A$  and  $B$ . In practice, point  $B$  and  $C$  are selected to be the same point. This simplifies the theorem somewhat: The tangential deviation at  $B$  to the tangent line coming from  $A$  is equal to the moment about  $B$  of the  $M/EI$  diagram between  $A$  and  $B$ .

While the order of the subscripts of  $\theta_{AB}$  in Eq. (5) is insignificant, the order of the subscripts of the tangential deviation  $t_{BA}$  in Eq. (8) is important. The first subscript is the location where the tangential deviation is measured. The second subscript is the point where the tangent line is drawn from.

## Analysis Procedure

Although the identification of rotations and tangential deviations is different for each structure, the following general procedure is suggested:

1. Establish the bending moment diagram by some other method.
2. Draw the  $M/EI$  diagram by dividing all ordinates of the bending moment diagram by the value of  $EI$  for each member.
3. Sketch the deflected shape of the structure in an approximate manner. The moment diagram provides assistance when doing this. The bending moment diagram tells which way the member curves. For bending moment diagrams drawn on the tension side, draw a “smiley” curve within the diagram with the corners of the smile closest to the member to show which way it curves.
4. Based on the deflected shape of the structure, draw tangent lines from key locations to identify rotations and tangential deviations. This is somewhat of an art because every structure is solved differently. However, practicing this art makes it easier over time, and fosters insight into the deformation of structures.

Moment-area analysis involves many area computations. To make this work easier, an auxiliary table with properties of various area shapes is provided elsewhere in these structural analysis notes.