

Load Combination

This document describes the load coincidence method (Wen 1990) and several load combination rules that are utilized in practice. The load coincidence method employs Poisson pulse processes, which are described in another document.

Load Coincidence Method

To understand the load coincidence method, three processes are considered. The two processes $s_1(t)$ and $s_2(t)$ are ordinary pulse processes that are either intermittent or always on. The third process is the “coincidence process” $s_{12}(t)=s_1(t)+s_2(t)$, which occur only when $s_1(t)$ and $s_2(t)$ coincide. It is stressed that $s_{12}(t)$ is only “on” when $s_1(t)$ and $s_2(t)$ coincide; otherwise it is zero, even though its intensity is defined by $s_{12}(t)=s_1(t)+s_2(t)$. Of primary interest in engineering applications is the probability that the intensity exceeds a threshold, r . Exceedance can happen in three ways: either $s_1(t)$ or $s_2(t)$ or $s_{12}(t)$ exceeds r . To address these three situations, three random variables are defined:

- $R_1=(\max\{s_1(t)\})$ for $t \in (0, T)$
- $R_2=(\max\{s_2(t)\})$ for $t \in (0, T)$
- $R_{12}=(\max\{s_{12}(t)\})$ for $t \in (0, T)$

The sought result is essentially the probability that the realization of any of these random variables exceed r :

$$p_f = P(R_1 > r \cup R_2 > r \cup R_{12} > r) \quad (1)$$

However, in keeping with the document on pulse processes, it is preferred to work with the complementary event, i.e., the one that is expressed in terms of the CDF for the lifetime maximum intensity, which by de Morgan’s rules is written:

$$\begin{aligned} F(r) &= 1 - p_f \\ &= 1 - P(R_1 > r \cup R_2 > r \cup R_{12} > r) \\ &= P(R_1 \leq r \cap R_2 \leq r \cap R_{12} \leq r) \end{aligned} \quad (2)$$

Although the three events are statistically dependent because R_{12} is positively correlated with both R_1 and R_2 , independence is assumed to simplify the subsequent derivations. Clearly, this introduces an error, but the error is conservative (it leads to an overestimation of the probability of threshold exceedance) and studies have shown that the error is typically not large (Wen 1990). Independence yields:

$$\begin{aligned} F(r) &= P(R_1 \leq r) \cdot P(R_2 \leq r) \cdot P(R_{12} \leq r) \\ &= F_{R_1}(r) \cdot F_{R_2}(r) \cdot F_{R_{12}}(r) \end{aligned} \quad (3)$$

The first two factors in the right-hand side are the CDF for the lifetime maximum value of the two processes $s_1(t)$ and $s_2(t)$. Expressions for these are provided in the document on pulse processes. The CDF for the lifetime maximum value of $s_{12}(t)$ is the subject of the following derivations. As a starting point it is recognized that, during any short time

interval, Δt , coincidence between $s_1(t)$ and $s_2(t)$ can occur in two mutually exclusive ways. The first is that $s_1(t)$ is “on” during Δt and $s_2(t)$ comes on during the occurrence of $s_1(t)$. The duration of the occurrence of $s_1(t)$ is here denoted by d_1 . According to the Poisson distribution, the probability of occurrence of $s_1(t)$ in the interval Δt is

$$P(x > 0) = 1 - p(0) = 1 - \exp(-\lambda_1 \cdot \Delta t) \quad (4)$$

where λ_1 is the rate of occurrences with non-zero occurrences. Similarly, the probability that $s_2(t)$ comes on during d_1 is

$$P(x > 0) = 1 - \exp(-\lambda_2 \cdot d_1) \quad (5)$$

Therefore, the probability of this first source of coincidence events is:

$$\begin{aligned} P(\text{Coincidence, Case 1}) &= (1 - \exp(-\lambda_1 \cdot \Delta t)) \cdot (1 - \exp(-\lambda_2 \cdot d_1)) \\ &\approx \lambda_1 \cdot \Delta t \cdot (1 - \exp(-\lambda_2 \cdot d_1)) \end{aligned} \quad (6)$$

where the second equality is made under the assumption that Δt is indeed small. The other possibility for coincidence is that $s_2(t)$ is “on” during Δt and that $s_1(t)$ comes on during the occurrence of $s_2(t)$. By denoting the duration of the occurrence of $s_2(t)$ by d_2 , the following result analogous to Eq. (6) is obtained:

$$P(\text{Coincidence, Case 2}) \approx \lambda_2 \cdot \Delta t \cdot (1 - \exp(-\lambda_1 \cdot d_2)) \quad (7)$$

Because the two possibilities are mutually exclusive, the final probability of coincidence is obtained by simple summation:

$$P(\text{Coincidence}) \approx \lambda_1 \cdot \Delta t \cdot (1 - \exp(-\lambda_2 \cdot d_1)) + \lambda_2 \cdot \Delta t \cdot (1 - \exp(-\lambda_1 \cdot d_2)) \quad (8)$$

Furthermore, because the probability of more than one coincidence in Δt is negligible, the average number of coincidences in Δt equals $P(\text{Coincidence})$. Dividing this number by the time interval yields the rate of coincidence:

$$\lambda_{12} = \frac{P(\text{Coincidence})}{\Delta t} \approx \lambda_1 \cdot (1 - \exp(-\lambda_2 \cdot d_1)) + \lambda_2 \cdot (1 - \exp(-\lambda_1 \cdot d_2)) \quad (9)$$

This important result is further simplified by noting that:

$$\lambda_{12} \leq \lambda_1 \cdot \lambda_2 \cdot (d_1 + d_2) \quad (10)$$

However, d_1 and d_2 are random variables, and this is addressed by utilizing the expectation:

$$\lambda_{12} = \int \int \lambda_1 \lambda_2 (d_1 + d_2) \cdot f(d_1) \cdot f(d_2) dd_1 dd_2 = \lambda_1 \lambda_2 (\mu_{d_1} + \mu_{d_2}) \quad (11)$$

This expression has been verified by Monte Carlo sampling analysis and has proven accurate even when $\lambda \mu_d$ is not very small for either of the processes (Wen 1990). Having the occurrence rate for the coincidence process, its mean duration is obtained by the following reasoning: The probability that the coincidence process is “on” is $\lambda_{12} \mu_{d_{21}}$ and this is equal to the probability that both of the individual processes are “on, namely $(\lambda_1 \mu_{d_1})(\lambda_2 \mu_{d_2})$. As a result:

$$\mu_{d12} = \frac{\lambda_1 \lambda_2 \mu_{d1} \mu_{d2}}{\lambda_{12}} \quad (12)$$

Combination of Eqs. (11) and (12) yields:

$$\mu_{d12} = \frac{\mu_{d1} \mu_{d2}}{\mu_{d1} + \mu_{d2}} \quad (13)$$

The third and final quantity required to describe the coincidence process, $s_{12}(t)$, is its arbitrary point in time (APIT) distribution. Essentially, the techniques from the topic functions of random variables are employed to obtain this distribution, i.e., the distribution of $s_{12}(t) = s_1(t) + s_2(t)$. In doing so, $F_{Y1}(r)$ and $F_{Y2}(r)$ are given and $F_{Y12}(r)$ is sought. Having determined all three characteristics of the coincidence process, the CDF for its maximum lifetime intensity is, from the document on pulse processes:

$$F_{R_{12}} = \exp\left(-\lambda_{12} \cdot T \cdot (1 - F_{Y_{12}}(r))\right) \quad (14)$$

Consequently, the result sought in this document, expressed earlier in Eq. (3), is

$$\begin{aligned} F(r) &\approx F_{R_1}(r) \cdot F_{R_2}(r) \cdot F_{R_{12}}(r) \\ &\approx \exp\left(-\lambda_1 T (1 - F_{Y_1}(r)) - \lambda_2 T (1 - F_{Y_2}(r)) - \lambda_{12} T (1 - F_{Y_{12}}(r))\right) \end{aligned} \quad (15)$$

In situations where one load is “always on” this expression simplifies. For example, if $s_1(t)$ is always on, then $s_2(t)$ cannot occur alone, and Eq. (15) simplifies to:

$$F(r) \approx \exp\left(-\lambda_1 T (1 - F_{Y_1}(r)) - \lambda_{12} T (1 - F_{Y_{12}}(r))\right) \quad (16)$$

Load Reduction Factor Rule

The objective in the previous derivations for the load coincidence method was to compute the probability that the overall process $\max\{s_1(t) + s_2(t)\}$ does or does not exceed r . Importantly, $\max\{s_1(t) + s_2(t)\}$ should not be confused with the more limited process $s_{12}(t)$, which occur only when $s_1(t)$ and $s_2(t)$ coincide. Simplified load combination rules, such as the load reduction factor rule, addresses the same problem but without employing the coincidence process $s_{12}(t)$ or its lifetime maximum value R_{12} . Instead, the load combination rules only assume that the lifetime maximum value of the individual processes, i.e., R_1, R_2, R_3 , etc. are available. Accordingly, design criteria are formulated in terms of R_1, R_2, R_3 , etc. rather than in terms of $\max\{s_1(t) + s_2(t)\}$. If there were no chance of coincidence the rule would be simple: $\max\{s_1(t) + s_2(t)\} = \max\{R_1 + R_2\}$. To account for the possibility of coincidence, the design criteria usually involve some auxiliary load factors. For example, the load reduction factor rule provides the following approximation for two load processes

$$\max\{s_1(t) + s_2(t)\} \approx \max\{R_1, R_2, \gamma_1(R_1 + R_2)\} \quad (17)$$

where typical values for γ_1 are in the range 0.7 to 0.8. The borderline between what is considered exceedance of $\max\{s_1(t)+s_2(t)\}$ and what is considered non-exceedance is shown by dashed lines in Figure 1. For three load processes the rule generalizes to

$$\max\{s_1(t)+s_2(t)+s_3(t)\} \approx \max \left\{ \begin{array}{l} R_1, R_2, R_3, \gamma_1(R_1+R_2), \\ \gamma_1(R_1+R_3), \gamma_1(R_2+R_3), \\ \gamma_2(R_1+R_2+R_3) \end{array} \right\} \quad (18)$$

where typically $\gamma_2=0.66$.

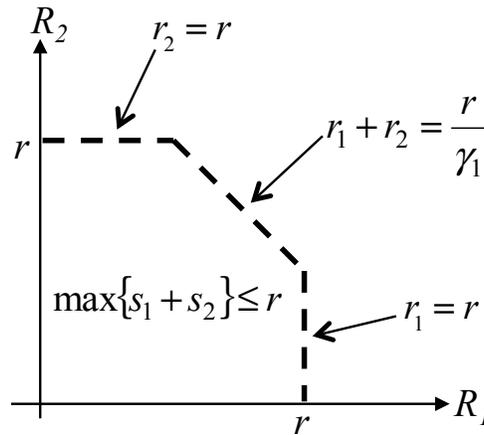


Figure 1: Visualization of the failure domain for the LRF rule.

Square Root of Sum of Squares Rule

With this load combination rule, exceedance of \max above r is assumed to occur when the following quantity exceeds r :

$$\max\{s_1(t)+s_2(t)\} \approx \sqrt{R_1^2 + R_2^2} \quad (19)$$

The borderline between exceedance and non-exceedance is shown as a dashed line in Figure 2. For three loads, this rule generalizes to:

$$\max\{s_1(t)+s_2(t)+s_3(t)\} \approx \sqrt{R_1^2 + R_2^2 + R_3^2} \quad (20)$$

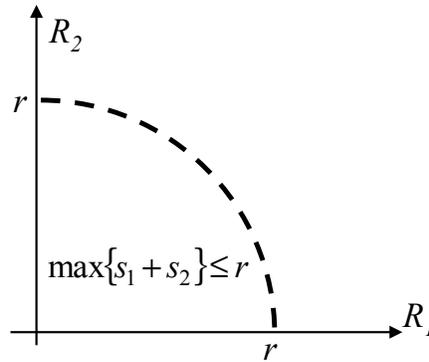


Figure 2: Visualization of the failure domain for the SRSS rule.

Companion Action Factor Rule

This rule is adopted in recent version of the National Building Code of Canada, where several load cases must be considered. Each load case has a primary load and one or more companion loads. In the case of two loads, the approximation is:

$$\max\{s_1(t) + s_2(t)\} \approx \max\{R_1 + \gamma_2 R_2, R_2 + \gamma_1 R_1\} \tag{21}$$

The borderline between exceedance and non-exceedance is shown as a dashed line in Figure 3. For three loads, this rule generalizes to:

$$\max\{s_1(t) + s_2(t) + s_3(t)\} \approx \max\left\{\begin{matrix} R_1 + \gamma_2 R_2 + \gamma_3 R_3, \\ R_2 + \gamma_1 R_1 + \gamma_3 R_3, \\ R_3 + \gamma_1 R_1 + \gamma_2 R_2 \end{matrix}\right\} \tag{22}$$

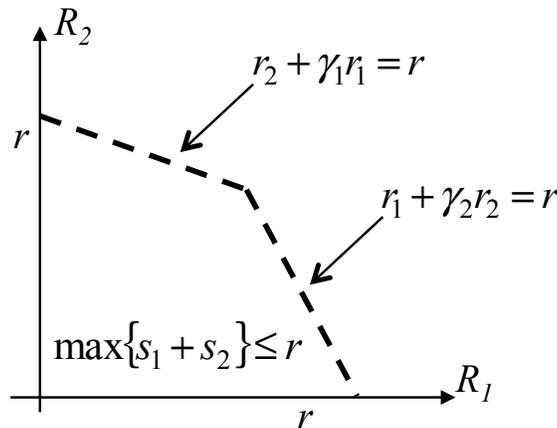


Figure 3: Visualization of the failure domain for the CAF rule.

Turkstra's Rule

Turkstra's load combination rule is unique in the sense that it introduces APIT values in the design criteria. As a result, this rule cannot be visualized in the R_1 - R_2 plane as the other rules earlier. For two loads, Turkstra's rule reads:

$$\max \{s_1(t) + s_2(t)\} \approx \max \{R_1 + s_2, R_2 + s_1\} \quad (23)$$

The borderline between exceedance and non-exceedance is shown as a dashed line in Figure 3. For three loads, this rule generalizes to:

$$\max \{s_1(t) + s_2(t) + s_3(t)\} \approx \max \{R_1 + s_2 + s_3, R_2 + s_1 + s_3, R_3 + s_2 + s_1\} \quad (24)$$

(More details to be written.)

References

Wen, Y. K. (1990). *Structural load modeling and combination for performance and safety evaluation*. Elsevier.