Lateral Torsional Buckling

The phenomenon called lateral torsional buckling is illustrated for a beam with an I-section in Figure 1. Suppose the beam is loaded with downward-acting distributed load, i.e., load in the negative z-direction. Unless the beam is restrained from displacing in the y-direction, the deformation shown in Figure 1 may occur. The phrase "lateral torsional buckling" is appropriate because the cross-section both rotates and displaces laterally. It is also a buckling phenomenon because the lateral stiffness is affected by the intensity of the load that is applied to the beam. In fact, the buckling load is determined by the lateral stiffness being zero.

Double-symmetric Cross-sections

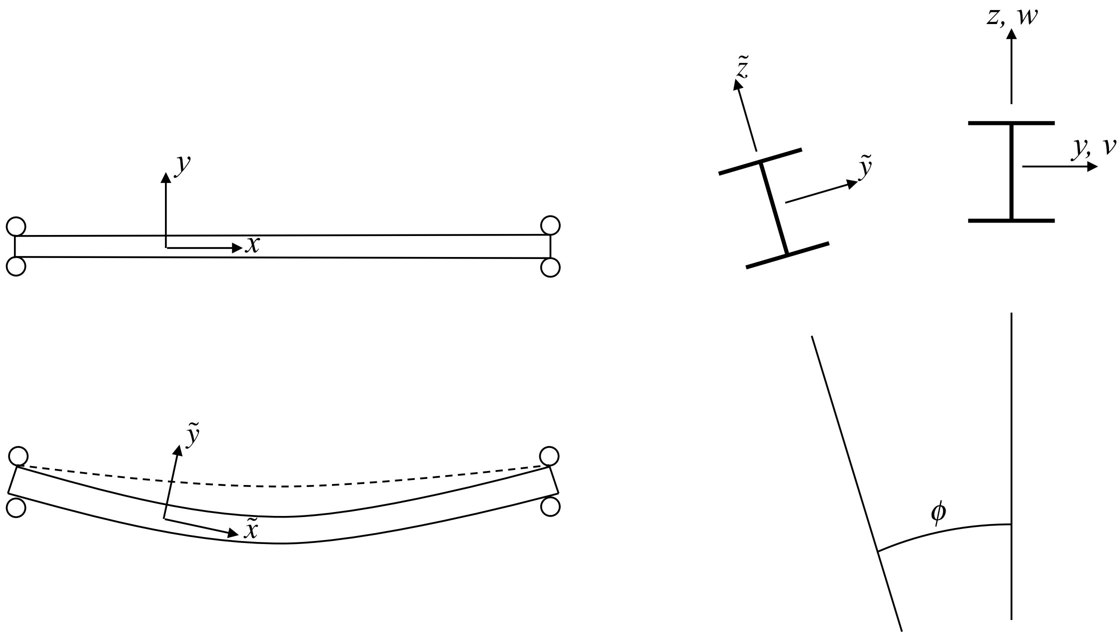

Figure 1: Lateral torsional buckling of I-section

Transformation between Original and Deformed Configuration

To account for the effect of the applied load on the lateral stiffness it is necessary to consider equilibrium in the displaced configuration. For that reason, a new coordinate system $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z})$ is established in the displaced configuration as shown in Figure 1. A point $\mathbf{x}$=(x, y, z) prior to deformation has the coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ after deformation. The new coordinate system represents a rotation of the original one, and a rotation matrix, $\mathbf{R}$, relates the two systems:

$$\tilde{\mathbf{x}} = \mathbf{R}\mathbf{x} \tag{1}$$

From the math-document on vectors and geometry it is understood that the rotation matrix contains "direction cosines:"

$$\tilde{x}_i = c_{ij} x_j \tag{2}$$

where c_{ij} is the cosine between the vector $\tilde{x}_i$ and x_j, i.e., the amount of $\tilde{x}_i$ in the direction of x_j. With reference to Figure 1, and provided small rotations so that cos(θ)≈1 and sin(θ)≈θ, the rotation matrix is:

$$\begin{Bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{Bmatrix} = \begin{bmatrix} 1 & -\theta_z & \theta_y \\ \theta_z & 1 & \phi \\ -\theta_y & -\phi & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{3}$$

Furthermore, for small deformations, $\theta_z = dv/dx$ and $\theta_y = -dw/dx$, and the rotation matrix reads:

$$\begin{Bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{dv}{dx} & -\frac{dw}{dx} \\ \frac{dv}{dx} & 1 & \phi \\ \frac{dw}{dx} & -\phi & 1 \end{bmatrix} \cdot \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{4}$$

Differential Equation

Given the transformation between the original and deformed configurations it is possible to determine the bending moments in the deformed beam due to a bending moment, M_o, in the un-deformed beam:

$$\begin{Bmatrix} \tilde{M}_x \\ \tilde{M}_y \\ \tilde{M}_z \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{dv}{dx} & -\frac{dw}{dx} \\ \frac{dv}{dx} & 1 & \phi \\ \frac{dw}{dx} & -\phi & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ M_y \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\frac{dv}{dx} \cdot M_y \\ M_y \\ -\phi \cdot M_y \end{Bmatrix} \tag{5}$$

For small deformations, the curvatures are approximately equal in the two coordinate systems, hence:

$$\kappa_{\tilde{y}} = \kappa_y = \frac{d^2 w}{dx^2} \tag{6}$$

$$\kappa_{\tilde{z}} = \kappa_z = \frac{d^2 v}{dx^2} \tag{7}$$

This means that the moment-curvature relationship for bending about the strong axis in the deformed configuration is:

$$M_{\tilde{y}} = M_y = -EI_y \frac{d^2 w}{dx^2} \tag{8}$$

which is the ordinary beam bending equation. The moment-curvature relationship for bending about the weak axis in the deformed configuration is:

$$M_{\tilde{z}} = -\phi \cdot M_y = EI_z \frac{d^2 v}{dx^2} \tag{9}$$

The relationship between torque and rotation is:

$$M_{\tilde{x}} = \frac{dv}{dx} \cdot M_y = GJ \cdot \frac{d\phi}{dx} - EC_w \cdot \frac{d^3\phi}{dx^3} \tag{10}$$

Differentiating Eq. (10) with respect to x and combining it with bending around the weak axis given by Eq. (9) yields:

$$GJ \cdot \phi'' - EC_w \cdot \phi'''' + \frac{M_y^2}{EI_z} \cdot \phi = 0 \tag{11}$$

Solution without Warping Torsion

First, consider a solution without warping torsion, in which case Eq. (11) simplifies to:

$$\phi'' + \left(\frac{M_y^2}{EI_z \cdot GJ} \right) \cdot \phi = 0 \tag{12}$$

The general solution to this differential equation is

$$\phi(x) = C_1 \cdot \cos\left(\sqrt{\frac{M_y^2}{EI_z \cdot GJ}} \cdot x \right) + C_2 \cdot \sin\left(\sqrt{\frac{M_y^2}{EI_z \cdot GJ}} \cdot x \right) \tag{13}$$

To gain insight into the lateral torsional buckling phenomenon, consider a simply supported beam with “fork”-type supports, i.e., ϕ=0 at both ends of the beam. This yields two equations to determine C_1 and C_2:

$$\phi(0) = C_1 = 0 \quad \Rightarrow \quad C_1 = 0 \tag{14}$$

$$\phi(L) = C_2 \cdot \sin\left(\sqrt{\frac{M_y^2}{EI_z \cdot GJ}} \cdot L \right) = 0 \tag{15}$$

From Eq. (15) it is understood that a non-trivial solution requires that:

$$\sqrt{\frac{M_y^2}{EI_z \cdot GJ}} \cdot L = n \cdot \pi \tag{16}$$

where n is any integer. Each integer value corresponds to a value of M_y. The smallest non-zero value of M_y is the critical value at which the beam buckles laterally:

$$M_{y,cr} = \frac{\pi}{L} \cdot \sqrt{EI_z \cdot GJ} \qquad (17)$$

Solution with Warping Torsion

Reconsider Eq. (11), now with warping torsion, and rewritten on the form:

$$\phi'''' - \underbrace{\frac{GJ}{EC_w}}_{\alpha} \cdot \phi'' - \underbrace{\frac{M_y^2}{EI_z \cdot EC_w}}_{\beta} \cdot \phi = 0 \qquad (18)$$

The general solution to this differential equation is:

$$\phi(x) = C_1 \cdot \cos(\gamma \cdot x) + C_2 \cdot \sin(\gamma \cdot x) + C_3 \cdot \cosh(\eta \cdot x) + C_4 \cdot \sinh(\eta \cdot x) \qquad (19)$$

where:

$$\gamma = \sqrt{-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta}} \qquad (20)$$

$$\eta = \sqrt{\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta}} \qquad (21)$$

Four boundary conditions are required to determine the four constants C_1, C_2, C_3, and C_4. For the simply supported beam, both ϕ and ϕ'' are zero at both ends of the beam. The latter implies that the bi-moment is zero at the ends, i.e., the beam is free to warp at the ends. Nevertheless, warping torsion contributes to the solution. In particular, to obtain a non-trivial solution:

$$\sin(\gamma \cdot L) = 0 \quad \Rightarrow \quad \gamma \cdot L = n \cdot \pi \qquad (22)$$

The smallest critical value of γ is obtained for n=1:

$$\sqrt{-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta}} \cdot L = \pi \qquad (23)$$

Solving for β in order to ultimately solve for M_y yields:

$$\beta = \left(\frac{\pi^2}{L^2} + \frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{4} = \frac{\pi^4}{L^4} + \frac{\pi^2}{L^2}\alpha \qquad (24)$$

Substitution of the expressions for α and β yields:

$$M_{y,cr} = \frac{\pi}{L} \cdot \sqrt{EI_z GJ + I_z C_w \frac{E^2\pi^2}{L^2}} \qquad (25)$$

Single-symmetric Cross-sections (The Wagner Effect)

(Yet to be written.)