LQI and SWTP

The LQI (life quality index) is an index that measures incremental changes in the health and safety of a population. If money spent on an infrastructure project yields an increase in the LQI then the resource allocation is meaningful, otherwise not. Although the LQI is philosophically interesting and has received significant attention in the structural reliability community, the translation into effective decision-making for individual structures remains an interesting challenge. Another challenge is that the benefit to society of human actions can arguably be measured in different ways. For example, reduction in the cumulative pain and suffering that individuals in a population experiences may be another indicator that should drive societal research allocation (Kahneman 2011). Thus, in the following derivations one should be mindful of the complexities associated with estimating what constitutes improvement in life quality. In fact, other documents on this website focus on direct consequence modelling and minimization of total cost or some other direct measure of utility.

Several research groups have contributed to the development of the LQI. The group centered at the University of Waterloo in Canada introduced the index in a seminal book (Nathwani et al. 1997) that was later revised and expanded (Nathwani et al. 2009). The same researchers discussed a broader set of economic risk acceptance criteria (Lind 2002) and provided a derivation of the LQI economic principles (Pandey et al. 2006). The Canadian group also introduced the SWTP concept, i.e., the societal willingness to pay (Pandey and Nathwani 2004). Research led by Professor Rackwitz at the Technical University of Munich applied the LQI in reliability-based optimal design (Rackwitz 2002). This group also addressed the influence of discounting and other factors (Rackwitz 2006), and applied the LQI to aging infrastructure (Rackwitz and Joanni 2009) and seismic risk mitigation (Sánchez-Silva and Rackwitz 2004). Research at the Technical University of Danmark led by Professor Ditlevsen discussed the LQI in the concext of decision-making under uncertainty (Ditlevsen 2003) and provided revisions of the original formulation (Ditlevsen 2004) that became a subject of debate (Rackwitz 2005). The Danish group also introduced the "life quality time allocation index" (Ditlevsen and Friis-Hansen 2005) and they provided a discussion of the Canadian research group's derivation of the LQI from economic principles (Ditlevsen and Friis-Hansen 2008).

The two key ingredients of the LQI are: 1) the life expectancy at birth, e, and 2) the real gross domestic product per person, g. The intensity of life quality is quantified by a function $f(g)$, and the duration of the good life is introduced by a function $h(t)$, where $t=(1-w)e$ is the time spent enjoying life, where w is the fraction of life spent in production of g rather than life enjoyment. The compound measure of life quality is

$$L = f(g) \cdot h(t) \tag{1}$$

The change in L, measured by the differential is, using the product rule of differentiation:

$$dL = \left(\frac{df}{dg} \cdot dg \right) \cdot h + f \cdot \left(\frac{dh}{dt} \cdot dt \right) \tag{2}$$

Normalizing by L yields:

$$\begin{aligned}\frac{dL}{L} &= \frac{1}{f\cdot h}\cdot\left(\frac{df}{dg}\cdot dg\right)\cdot h + \frac{1}{f\cdot h}\cdot f\cdot\left(\frac{dh}{dt}\cdot dt\right)\\ &= \frac{1}{f}\cdot\frac{df}{dg}\cdot dg + \frac{dh}{dt}\cdot\frac{1}{h}\cdot dt\end{aligned} \tag{3}$$

Multiplying the first term by g/g and the second term by t/t, defining the constants k_1 and k_2, and realizing that dt/t equals de/e yields:

$$\begin{aligned}\frac{dL}{L} &= \frac{g}{f}\frac{df}{dg}\cdot\frac{dg}{g} + \frac{dh}{dt}\cdot\frac{dt}{t}\cdot\frac{t}{h}\\ &= \underbrace{\left(\frac{g}{f}\frac{df}{dg}\right)}_{k_1}\cdot\frac{dg}{g} + \underbrace{\left(\frac{dh}{dt}\cdot\frac{t}{h}\right)}_{k_2}\cdot\frac{dt}{t}\\ &= k_1\cdot\frac{dg}{g} + k_2\cdot\frac{de}{e}\end{aligned} \tag{4}$$

The constants k_1 and k_2 quantify the relative impact of increasing g, i.e., improving the intensity of life enjoyment, and the life expectancy, e, i.e., the duration of life enjoyment. Assuming that the ratio k_1/k_2 remains constant implies that k_1 and k_2 are constants, which leads to the two equations:

$$\begin{aligned}k_1 &= \frac{g}{f}\frac{df}{dg}\\ k_2 &= \frac{t}{h}\cdot\frac{dh}{dt}\end{aligned} \tag{5}$$

These differential equations can be written in the following form:

$$\begin{aligned}\frac{df}{dg} - \frac{k_1}{g}\cdot f &= 0\\ \frac{dh}{dt} - \frac{k_2}{t}\cdot h &= 0\end{aligned} \tag{6}$$

These are ordinary linear homogeneous differential equations with variable coefficients. The general solution to such equations is described in a math document on this website, and the specific solutions are:

$$\begin{aligned}f(g) &= C_1\cdot e^{-\ln(g)\cdot k_1} = C_1\cdot g^{-k_1}\\ h(t) &= C_2\cdot e^{-\ln(t)\cdot k_2} = C_2\cdot t^{-k_2}\end{aligned} \tag{7}$$

In the product $f(g)h(t)$ the constants C_1 and C_2 are irrelevant, and renaming k_1 and k_2 yields (Nathwani et al. 1997):

$$\begin{aligned}f(g) &= g^q\\ h(t) &= t^s\end{aligned} \tag{8}$$

so that the LQI in Eq. (1) takes the form:

$$\begin{aligned} L &= g^q \cdot t^s \\ &= g^q \cdot \big((1-w)\cdot e\big)^s \end{aligned} \tag{9}$$

To obtain expressions for q and s it is postulated that the gross domestic product is proportional to the time spent in production. I.e., g=we so that

$$L = \big(w\cdot e\big)^q \cdot \big((1-w)\cdot e\big)^s \tag{10}$$

Assuming that L is in practice maximized by people balancing their time producing g and their leisure time, i.e., assuming that people live at the optimal fraction w yields:

$$\frac{dL}{dw} = 0 \tag{11}$$

The solution to Eq. (11) is

$$q = \left(\frac{w}{1-w}\right)\cdot s \tag{12}$$

Setting q=w and s=1-w satisfies Eq. (12). Substitution of these values into Eq. (9) yields:

$$L = g^w \cdot e^{1-w} \cdot (1-w)^{1-w} \tag{13}$$

The last factor is practically constant, leading to the following expression for the LQI:

$$L = g^w \cdot e^{1-w} \tag{14}$$

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