Differential Equations

In other documents on linear algebra and nonlinear equations the objective is to determine the value of unknown parameters. With differential equations the objective is different. Here, the objective is to determine an unknown function from an equation where the function's derivatives appear.

The first step in solving a differential equation is to identify its type. For example, if it is found that the problem contains a first-order ordinary linear homogeneous differential equation with constant coefficients then particular tools are available. Other tools are available for other types of equations.

The type of a differential equation is determined by its characteristic in each of the following five categories:

- **Ordinary vs. partial**
  - The equation is ordinary if the derivatives of the function are with respect to only one parameter, e.g., x. Conversely, when partial derivatives with respect to several parameters appear then the equation is partial.
- **Linear vs. nonlinear**
  - If the function and its derivative appear in linear form, i.e., without being, e.g., squared then the equation is linear.
- **Homogeneous vs. inhomogeneous**
  - If all the terms of the equation contain the function or its derivative then the equation is homogeneous, otherwise not.
- **Constant vs. variable coefficients**
  - If the coefficients that multiply the function and its derivatives contain any of the variables that the function depends on then the equation is said to have variable coefficients, otherwise not.
- **Order**
  - A differential equation is first-order, second-order, etc. The order is the highest order of derivative that appears in the equation.

Each type of equation has a specific solution strategy. The "map" in Figure 1 provides a guide to select the correct strategy in the following sections. Each letter at the bottom of Figure 1 refers to one section.

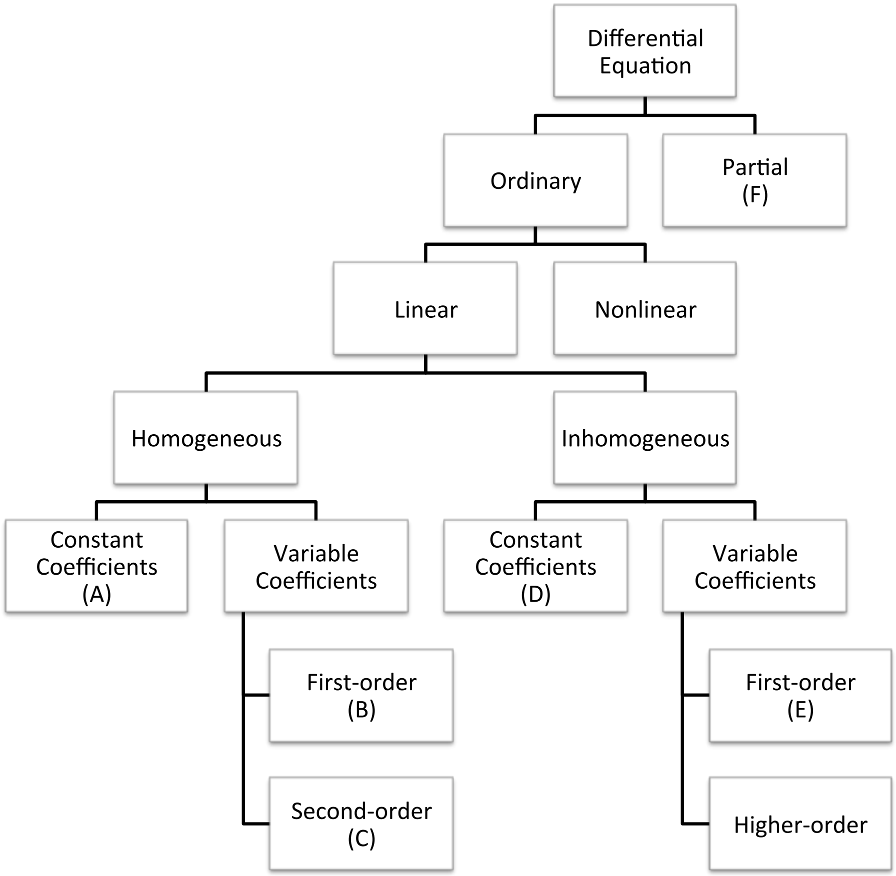

Figure 1: Identification chart for differential equations.

The following sections provide solutions for the problems that are identified in Figure 1. However, sometimes the problem is sufficiently simple that integration provides the solution. One case in point is the differential equation for beam bending:

$$w''''(x) = \frac{q(x)}{EI} \tag{1}$$

The homogeneous solution, i.e., the solution to the differential equation $w''''(x)=0$ is:

$$w_h(x) = C_1 \cdot x^3 + C_2 \cdot x^2 + C_3 \cdot x + C_4 \tag{2}$$

The particular solution, i.e., the solution that is associated with the right-hand side term depends on the variation of the load intensity $q(x)$. For uniform load $q(x)=q_0$ the particular solution is obtained by four times integration:

$$w_p(x) = \frac{q_0}{EI} \cdot x^4 \tag{3}$$

A

Consider differential equations of the form

$$\begin{aligned} f'(x) + a \cdot f(x) &= 0 \\ f''(x) + a \cdot f'(x) + b \cdot f(x) &= 0 \\ &\cdots \end{aligned} \tag{4}$$

where *a*, *b*, … are constants. The first step in the solution procedure is to establish the characteristic equation. It is defined as a polynomial in λ with order matching the order of the differential equation. For the problems in Eq. (4) the characteristic equations are

$$\begin{aligned} \lambda + a &= 0 \\ \lambda^2 + a \cdot \lambda + b &= 0 \\ &\cdots \end{aligned} \tag{5}$$

The solution depends upon the values of the roots of the characteristic equation. If the roots are real and distinct then the solution is

$$f_h(x) = C_1 \cdot e^{\lambda_1 \cdot x} + C_2 \cdot e^{\lambda_2 \cdot x} + C_3 \cdot e^{\lambda_3 \cdot x} + \cdots \tag{6}$$

where subscript *h* is employed throughout to indicate that it is the solution to a homogeneous equation. If the roots are real but equal then the solution is

$$f_h(x) = \left(C_1 + C_2 x + C_3 x^2 + \cdots\right) \cdot e^{\lambda \cdot x} \tag{7}$$

which for a second-order equation reads

$$f_h(x) = \left(C_1 + C_2 x\right) \cdot e^{-\frac{a}{2} \cdot x} \tag{8}$$

If the differential equation is second-order and the roots are complex conjugate numbers then the solution is

$$f_h(x) = e^{-\frac{a}{2} \cdot x} \left(C_1 \cdot \cos(\omega \cdot x) + C_2 \cdot \sin(\omega \cdot x)\right) \tag{9}$$

where

$$\omega = \sqrt{b - \frac{1}{4} a^2} \tag{10}$$

For higher-order differential equations, if the roots are complex and different, the solution is

$$f_h(x) = C_1 \cdot e^{\alpha_1 \cdot x} \cdot \cos(\omega_1 \cdot x) + C_2 \cdot e^{\alpha_1 \cdot x} \cdot \sin(\omega_1 \cdot x) + \cdots \text{(possible other pairs)} \tag{11}$$

where

$$\lambda_i = \alpha_i + i \cdot \omega_i \tag{12}$$

Conversely, for higher-order differential equations, if the roots are complex and equal, the solution is

$$\begin{aligned} f_h(x) = & C_1 \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_2 \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ & + C_3 \cdot x \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_4 \cdot x \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ & + C_5 \cdot x^2 \cdot e^{\alpha \cdot x} \cdot \cos(\omega \cdot x) + C_6 \cdot x^2 \cdot e^{\alpha \cdot x} \cdot \sin(\omega \cdot x) \\ & + \cdots \end{aligned} \tag{13}$$

B

Consider the differential equation

$$f'(x) + p(x) \cdot f(x) = 0 \tag{14}$$

where $p(x)$ is a known function. The solution is

$$f_h(x) = c \cdot e^{-h(x)} \tag{15}$$

where

$$h(x) = \int p(x)\,dx \tag{16}$$

C

In addition to the previous section, several solution techniques are available for differential equations with variable coefficients. They include series expansion methods, reduction of order and application of the previous section, and solution of the special differential equation called Euler-Cauchy.

D

For inhomogeneous differential equations the total solution is the sum of the homogeneous and particular solutions:

$$f(x) = f_h(x) + f_p(x) \tag{17}$$

More coming…

E

Consider Eq. (14) with right-hand side equal to $r(x)$. The total solution is

$$f(x) = e^{-h(x)} \cdot \left(\int \left(e^{h(x)} \cdot r(x)dx \right) + C \right) \tag{18}$$

where

$$h(x) = \int p(x)\,dx \tag{19}$$