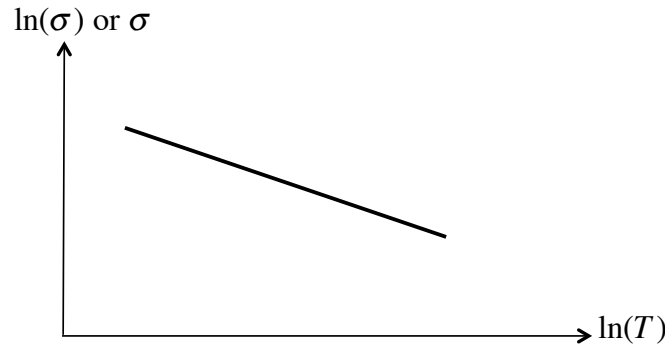


# Damage Accumulation Models

## A Creep Model

Creep is a long-term effect in structural materials, by which deformations slowly increase until possible failure. The phrase “duration-of-load” is often applied to the phenomenon because it captures the two key design variables of the problem: How much load is applied for how long. In fact, a simple visualization of the capacity of a material to sustain load over long time is shown in Figure 1. Failure due to creep is called “creep rupture” and Figure 1 is referred to as a “creep rupture curve.”



**Figure 1: Relationship between stress and time to creep rupture.**

Creep rupture curves like the one in Figure 1 are often entirely empirical, but there also exist damage accumulation models that model the phenomenon. One model is (Barrett and Foschi 1978)

$$\frac{d\alpha}{dt} = \begin{cases} a \cdot (\sigma - \sigma_0)^b \cdot \alpha^c & \text{if } \sigma > \sigma_0 \\ 0 & \text{if } \sigma \leq \sigma_0 \end{cases} \quad (1)$$

where  $\alpha$  is the measure of accumulated damage ( $\alpha=0$  denotes no damage while  $\alpha=1$  denotes rupture) and  $\sigma_0$  is the stress below which damage is assumed not to accumulate. The parameters  $a$ ,  $b$ , and  $c$  are calibration constants. In the following, consider the case  $\sigma > \sigma_0$  and rearrange Eq. (1) to the form:

$$\alpha^{-c} \cdot \frac{d\alpha}{dt} = a \cdot (\sigma - \sigma_0)^b \quad (2)$$

Next, this equation is integrated from  $t=0$  to  $t=T$ . To accomplish this it is necessary to apply integration by substitution to the left-hand side, which because  $\alpha(0)=0$  evaluates to:

$$\int_0^T \alpha^{-c} \cdot \frac{d\alpha}{dt} dt = \int_{\alpha(0)}^{\alpha(T)} x^{-c} dx = \frac{1}{1-c} \cdot \alpha(T)^{(1-c)} \quad (3)$$

This left-hand side together with the integral of the right-hand side yields the following solution to Eq. (2), i.e., the damage,  $\alpha(T)$ , at time  $T$ :

$$\frac{1}{1-c} \cdot \alpha(T)^{(1-c)} = \int_0^T a \cdot (\sigma - \sigma_0)^b dt \quad (4)$$

Rearranging and acknowledging the dependence of the stress on time yields the following solution for the evolution of creep:

$$\alpha(T)^{(1-c)} = a \cdot (1-c) \cdot \int_0^T (\sigma(t) - \sigma_0)^b dt \quad (5)$$

Creep rupture occurs when  $\alpha=1$ . According to Eq. (5) this leads to the following equation from which the time,  $T$ , until rupture can be solved:

$$1 = a \cdot (1-c) \cdot \int_0^T (\sigma(t) - \sigma_0)^b dt \quad (6)$$

For example, if the stress is constant with time then the time until creep rupture is:

$$T = \frac{1}{a \cdot (1-c) \cdot (\sigma - \sigma_0)^b} \quad (7)$$

Another damage accumulation model for creep is (Barrett and Foschi 1978):

$$\frac{d\alpha}{dt} = \begin{cases} a \cdot (\sigma - \sigma_0)^b + \lambda \cdot \alpha & \text{if } \sigma > \sigma_0 \\ 0 & \text{if } \sigma \leq \sigma_0 \end{cases} \quad (8)$$

Again the case  $\sigma > \sigma_0$  is considered and the model is an ordinary linear inhomogeneous differential equation of the first order:

$$\frac{d\alpha}{dt} - \lambda \cdot \alpha = a \cdot (\sigma - \sigma_0)^b \quad (9)$$

To solve this differential equation its characteristic equation is established:

$$\gamma - \lambda = 0 \quad (10)$$

The solution is  $\lambda$  and the homogeneous solution is:

$$\alpha(t) = C_1 \cdot e^{\lambda t} \quad (11)$$

Together with the particular solution, the damage evolution is described by:

$$\alpha(t) = e^{\lambda t} \cdot \int_0^T a \cdot (\sigma(t) - \sigma_0)^b \cdot e^{-\lambda t} dt \quad (12)$$

## A Crack Growth Model

One model for fatigue crack propagation is (Joint Committee on Structural Safety 2001)

$$\frac{da}{dn} = C \cdot Y(a) \cdot (\Delta S(n) \cdot \sqrt{\pi \cdot a})^m \quad (13)$$

where  $a(n)$  is the crack size,  $n$  is number of cycles,  $C$  is a material constant,  $Y(a)$  is a geometry-dependent function,  $\Delta S(n)$  is the stress range, and  $m$  is another material constant. Rearranging yields

$$a^{-\frac{m}{2}} \cdot \frac{da}{dn} = C \cdot Y(a) \cdot (\Delta S(n) \cdot \sqrt{\pi})^m \quad (14)$$

To integrate this equation from 0 to  $N$ , integration by substitution is first applied to the left-hand side:

$$\int_0^N a^{-\frac{m}{2}} \cdot \frac{da}{dn} dn = \int_{a(0)}^{a(N)} x^{-\frac{m}{2}} dx = \frac{2}{2-m} \cdot a(N)^{(2-m)/2} \quad (15)$$

where it was assumed that  $a(0)=0$ . Integration of the right-hand side of Eq. (14), assuming constant stress range and constant geometry factor, yields:

$$\int_0^N C \cdot Y(a) \cdot (\Delta S(n) \cdot \sqrt{\pi})^m dn = C \cdot Y \cdot \Delta S^m \cdot \pi^{m/2} \cdot N \quad (16)$$

Combining the left-hand side and the right-hand obtained above yields the following integrated solution to Eq. (14), i.e., the crack size,  $a(N)$ , at cycle  $N$ :

$$\frac{2}{2-m} \cdot a(N)^{(2-m)/2} = C \cdot Y \cdot \Delta S^m \cdot \pi^{m/2} \cdot N \quad (17)$$

Suppose failure occurs if the crack size exceeds a critical threshold,  $a_{cr}$ , Eq. (17) can be rearranged to express failure as follows:

$$\frac{1}{C \cdot Y} \cdot \pi^{-m/2} \cdot \frac{2}{2-m} \cdot a(N)^{(2-m)/2} \leq \Delta S^m \cdot N \quad (18)$$

Another model for crack growth, formulated as a function of time instead of number of cycles, is

$$\frac{da}{dt} = A \cdot K^m(t) \quad (19)$$

where

$$K(t) = \sigma(t) \cdot \sqrt{a} \quad (20)$$

Following the solution procedure above, rearranging yields:

$$a^{-m/2} \cdot \frac{da}{dt} = A \cdot \sigma^m \quad (21)$$

Integration with respect to time from 0 to  $T$ , with application of integration by substitution to the left-hand side, yields:

$$\frac{2}{2-m} \cdot a(T)^{(2-m)/2} = A \cdot \int_0^T \sigma(t)^m dt \quad (22)$$

from which the crack size,  $a$ , can be solved for.

## References

Barrett, J. D., and Foschi, R. O. (1978). "Duration of load and probability of failure in wood. Part I. Modelling creep rupture." *Canadian Journal of Civil Engineering*, 5(4), 505–514.

Joint Committee on Structural Safety. (2001). *Probabilistic Model Code*.