Conjugate Beam Method

This is a method useful for the calculation of rotations and displacements of horizontal beams with arbitrary loads and support conditions. The basic version of this method, which is applicable only to simply supported beams, is called the "elastic load method." The generalized version is called the conjugate beam method, but even this method is somewhat rarely used in practice because the moment-area method and the unit virtual load method are more general and often easier to use. Nonetheless, the conjugate beam method provides interesting insight into the relationship between the "curvature diagram" and the displacement and rotation of beams. One starting point for understanding the method is to compare equations from the Euler-Bernoulli beam theory. First consider the similarity between these two equations:

$$q = \frac{dV}{dx} \qquad \Leftrightarrow \qquad \frac{M}{EI} = \frac{d\theta}{dx} \tag{1}$$

They show that if *M*/*EI* is placed as distributed load, *q*, on a beam, then the resulting shear force, *V*, is the rotation, θ, of the original beam. Similarly, the equations

$$q = \frac{d^2M}{dx^2} \qquad \Leftrightarrow \qquad \frac{M}{EI} = \frac{d^2w}{dx^2} \tag{2}$$

show that if *M*/*EI* is placed as distributed load, *q*, on a beam, then the resulting bending moment, *M*, is the displacement, *w*, of the original beam. This insight leads to the following procedure for computing rotations and displacements in simply supported beams: First, divide all ordinates of the bending moment diagram by the bending stiffness to produce the *M*/*EI* diagram, i.e., the curvature diagram. Then, the beam loaded by the *M*/*EI* diagram has the following properties:

- The shear force equals the rotation in the original beam
- The bending moment equals the deflection the original beam

To make this approach valid beyond simply supported beams it is necessary to modify the supports. That will be addressed shortly, but another justification of the approach is first provided. Consider the beam in Figure 1, which has a point rotation, θ, at a distance *x* from the left support. An expression for the rotation at A, in terms of θ, is:

$$\theta_A = \frac{t_{CB}}{L} = \frac{\theta \cdot (L-x)}{L} \tag{3}$$

Then the displacement at B is

$$\Delta_B = \theta_A \cdot x = \frac{\theta \cdot (L-x) \cdot x}{L} \tag{4}$$

and the rotation at C is

$$\theta_C = \frac{\Delta_B}{(L-x)} = \frac{\theta \cdot x}{L} \tag{5}$$

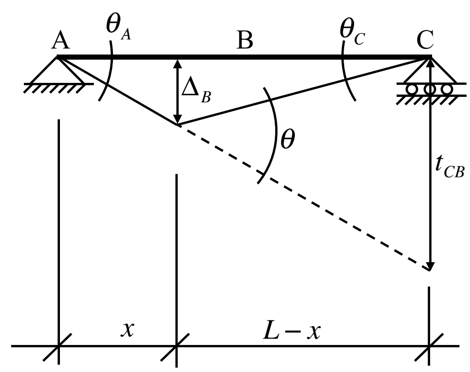

Figure 1: Simple supported beam with point rotation.

The expressions for θ_A, Δ_B, and θ_C in Eqs. (3), (4), and (5) will now be compared to another case, namely the simply supported beam loaded with a point load, shown in Figure 2. The analysis of this statically determinate problem yields the shear force and bending moment diagrams shown in Figure 2. If P is replaced by θ then these values are exactly the same as the values in Eqs. (3), (4), and (5). This shows that V and M in a beam loaded by θ, equals the rotation and displacement, respectively, of the beam subjected to the rotation θ. This generalizes this to many small rotations, $d\theta$.

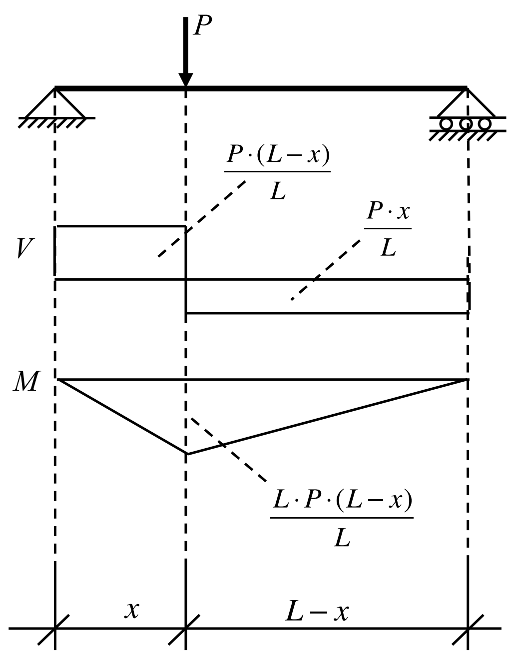

Figure 2: Simple supported beam with point load.

To obtain the general conjugate beam method it is necessary to modify the support reactions of the original beam before analyzing it with M/EI as loading. This is because supports with zero rotation and/or zero displacement in the original beam must have zero shear force and/or zero bending moment in the conjugate case. Figure 3 shows the translation of the most common support reactions or hinge conditions. Notice that conjugate beams translated from indeterminate original beams may be unstable, but that the M/EI loading acts to keep it stable.

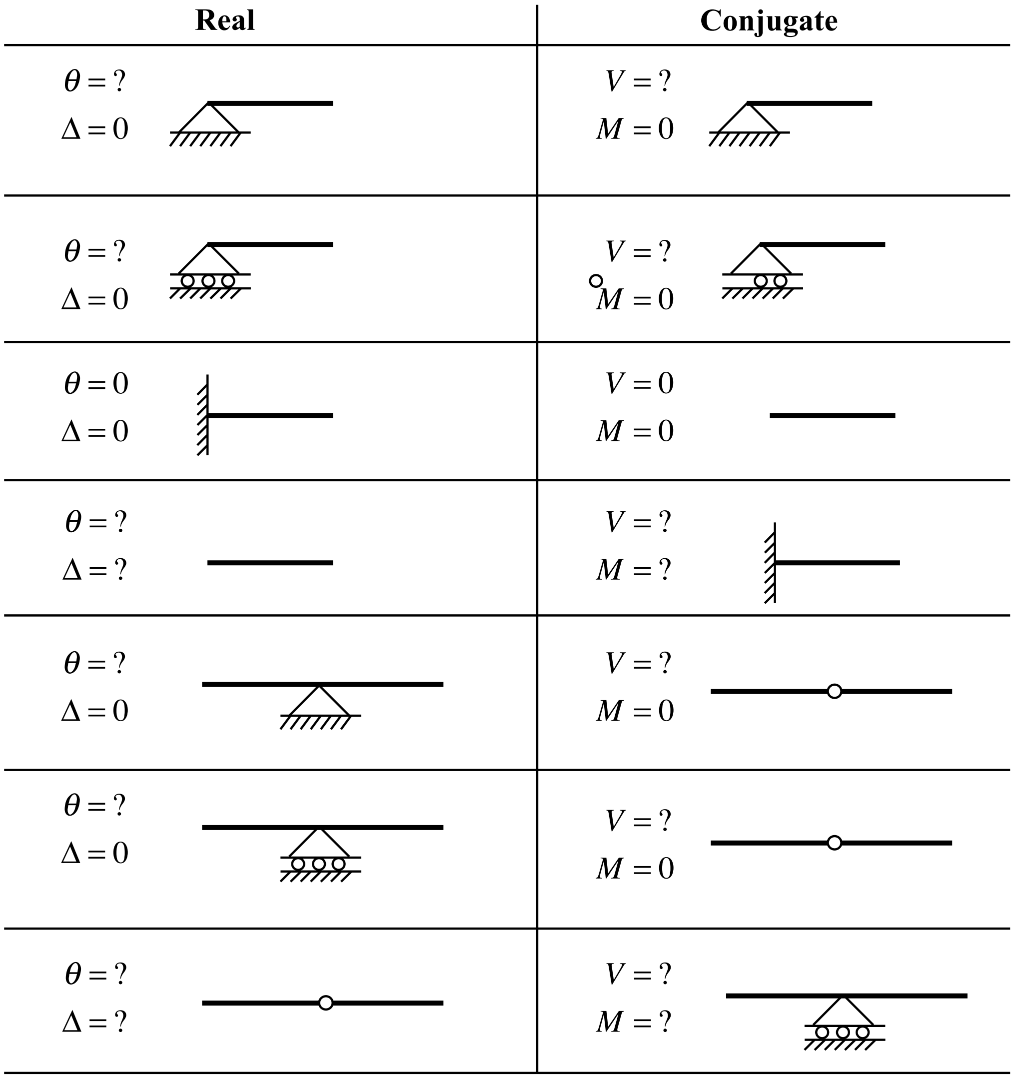

Figure 3: Conjugate support conditions.