Beams with Axial Force

In this document the Euler-Bernoulli beam theory is extended with P-delta effects, i.e., the influence of axial force on the bending stiffness. For this purpose, consider a member with constant axial force, *P*, which is positive in compression. To include the effect of the axial force, equilibrium is considered in the displaced configuration. For the infinitesimally small beam element in Figure 1, moment equilibrium about the rightmost edge at the neutral axis yields:

$$V \cdot dx + q_z \cdot dx \cdot \frac{dx}{2} - dM - P \cdot dw = 0 \tag{1}$$

The square of the infinitesimally small length *dx* is a higher-order effect that is neglected, so that rearranging Eq. (1) yields:

$$V = \frac{dM}{dx} + P \cdot \frac{dw}{dx} \tag{2}$$

The other equilibrium equation, as well as section integration, material law, and kinematic equations remain the same as in the ordinary beam theory. Thus, the modified differentiation equation for beam bending, now including P-delta effects, reads:

$$\begin{aligned} q_z &= \frac{dV}{dx} = \frac{d^2M}{dx^2} + P \cdot \frac{d^2w}{dx^2} = -\frac{d^2}{dx^2}\int_A \sigma \cdot z \ \mathrm{d}A + P \cdot \frac{d^2w}{dx^2} \\ &= -\frac{d^2}{dx^2}\int_A E \cdot \varepsilon \cdot z \ \mathrm{d}A + P \cdot \frac{d^2w}{dx^2} = \frac{d^2}{dx^2}\int_A E \cdot \frac{d^2w}{dx^2} \cdot z^2 \ \mathrm{d}A + P \cdot \frac{d^2w}{dx^2} \\ &= EI \cdot \frac{d^4w}{dx^4} + P \cdot \frac{d^2w}{dx^2} \end{aligned} \tag{3}$$

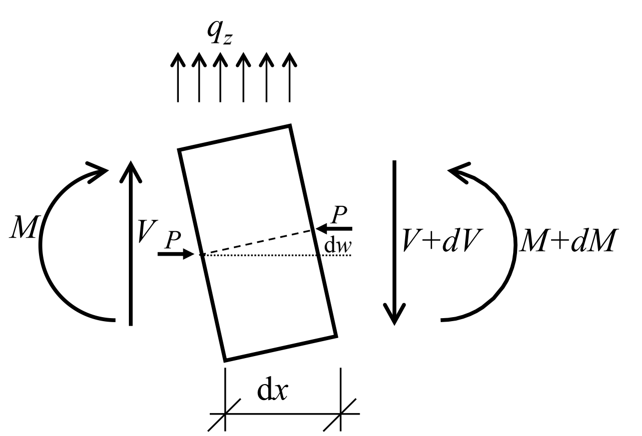

Figure 1: Equilibrium in the displaced configuration.

In the following, the homogeneous version of this equation is solved to get the Euler buckling loads under the assumption that *P* is positive, i.e., compressive. The characteristic equation is:

$$\gamma^4 + \frac{P}{EI} \cdot \gamma^2 = 0 \tag{4}$$

Solving the characteristic equation gives the following general solution:

$$w(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right) + C_3 \cdot x + C_4 \tag{5}$$

where the coefficients, C_i, are determined by boundary conditions that are unique to each beam case. In the case of tension, i.e., $P<0$, sin and cos are replaced by sinh and cosh. For the simply supported beam the boundary conditions are

$$w(0) = C_2 + C_4 = 0 \tag{6}$$

$$w(L) = C_1 \sin\left(\sqrt{\frac{P}{EI}} \cdot L\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}} \cdot L\right) + C_3 \cdot L + C_4 = 0 \tag{7}$$

$$w''(0) = -C_2 \frac{P}{EI} = 0 \tag{8}$$

$$w''(L) = -C_1 \frac{P}{EI} \sin\left(\sqrt{\frac{P}{EI}} \cdot L\right) - C_2 \frac{P}{EI} \cos\left(\sqrt{\frac{P}{EI}} \cdot L\right) = 0 \tag{9}$$

The only possibility of a non-trivial solution is

$$\sin\left(\sqrt{\frac{P}{EI}} \cdot L\right) = 0 \tag{10}$$

The sine function is zero when its argument is some integer multiple of π:

$$\sqrt{\frac{P}{EI}} \cdot L = n\pi \tag{11}$$

Solving for P gives the critical load:

$$P_{cr} = n^2 \cdot \frac{\pi^2 EI}{L^2} \tag{12}$$

The smallest buckling load, obtained for n=1, is the relevant buckling load. The displaced shape for n=1 is a quarter sine wave, with undetermined amplitude C_1 and $C_2=C_3=C_4=0$. For a higher integer value of n the buckling load is higher and the displaced shape has more curves.

The buckling load for beam-column elements with other boundary conditions is often written

$$P_{cr} = \frac{\pi^2 EI}{L_{cr}^2} \tag{13}$$

where L_{cr} is the length of a half sine wave identified within the displaced shape of the member. A few cases are listed here:

- $L_{cr}=L$ for a simply supported column of length L
- $L_{cr}=2L$ for a cantilevered column of length L
- $L_{cr}=0.5L$ for a fixed-fixed column of length L
- $L_{cr}\approx 0.7L$ for a fixed-pinned column of length L

Slenderness and Maximum Stress

A popular visualization of the potential for buckling is a diagram with "slenderness" on the abscissa axis and stress on the ordinate axis. Slenderness is first defined. Let the radius of gyration, i, of the cross-section be defined as

$$i=\sqrt{\frac{I}{A}} \qquad \Leftrightarrow \qquad I=i^2\cdot A \tag{14}$$

where I is the moment of inertia and A is the cross-sectional area. Next, define the slenderness ratio as

$$\lambda=\frac{L_{cr}}{i} \tag{15}$$

where L_{cr} is the length defined in Eq. (13). Now the stress in the cross-section at the buckling load is expressed

$$\sigma_{cr}=\frac{P_{cr}}{A}=\frac{\pi^2\cdot E\cdot I}{A\cdot L_{cr}^2}=\frac{\pi^2\cdot E\cdot i^2}{L_{cr}^2}=\frac{\pi^2\cdot E}{\lambda^2} \tag{16}$$

A schematic plot of σ_{cr} versus λ is shown in Figure 2. The curved line represents Eq. (16) and shows that the maximum allowable stress goes down as the slenderness increases. This is reasonable; a slender column can sustain less axial load than a column that either is shorter or has a more substantial cross-section. Figure 2 also indicates a cut-off at σ_y, which is a stress that cannot be exceeded under any circumstances.

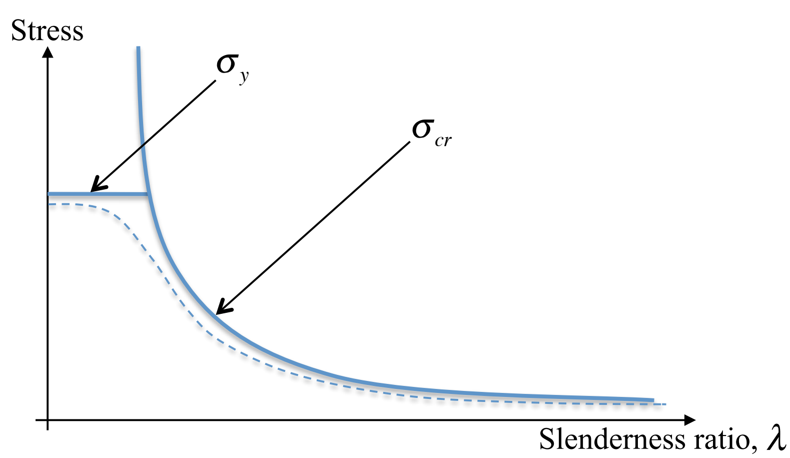

Figure 2: Slenderness and maximum stress.