Assumptions, Theories, Methods

Structural analysis is founded on the continuum assumption. Instead of considering the molecular composition of the material, the assumption is made that the material is a homogeneous continuum. Even when different materials are combined into a composite structure, such as in a bridge with a reinforced concrete slab that is attached to the steel girders underneath, it is assumed that each material agrees with the continuum assumption. While materials such as steel may be close to satisfying this hypothesis it is obvious that materials like concrete and timber have stronger variability. Nevertheless, this does not prevent a useful, powerful, and general-purpose structural analysis theory to be based on the continuum assumption. In fact, the entire field of structural engineering, as well as these notes, is based on the continuum assumption.

Although the resulting theory is powerful, it is important to recognize that structural analysis is not based on mechanics alone. Rather, the theory is formulated in terms of parameters that must be determined by experiment rather than theory. The modulus of elasticity and the yield stress are two examples. On one hand it is appealing that the fundamental parameters are measurable and have physical meaning. On the other hand, it should be duly noted that the laboratory experiments are what make the theory useful. Importantly, even when the continuum assumption is violated the experiments provide proxy values for the parameters of the theory. For example, wood beams with knots and imperfections are subjected to bending tests that provides the value of the maximum allowable bending stress, although this stress may not actually exist anywhere in the beam.

Additional assumptions are made within the theory of each type of structural members. They are usually related to the material behaviour and the deformation kinematics. In Euler-Bernoulli beam theory, for example, the assumptions are made that the material is linearly elastic according to Hooke’s law and that plane sections remain plane and perpendicular to the neutral axis during bending according to Navier’s hypothesis. Regardless of assumption and model, the keen reader is encouraged to continually question to the validity of existing theories and models and join the continuous quest for improved ones.

Problem Formulation

The objective in structural analysis is to determine the internal forces and deformations in the structure due to applied loads or deformations. Specifically, the objective is to find any or all of the following quantities:

- Section forces:
    - Axial force
    - Bending moment
    - Shear force
    - Torque
- Deformations:

    - Displacement
    - Rotation
- Stresses:
    - Axial stress
    - Shear stress
- Strains:
    - Axial strain
    - Shear strain

Additional responses may be sought in dynamic analysis, such as accelerations, and in nonlinear analysis, such as plastic deformations. In the context of linear static structural analysis, however, the list above is appropriate. Attention is now turned to how these responses are calculated. For this purpose, the quantities in the list above are linked in Figure 1 with the external loads on the structure. In particular, the external load is related to the stress resultants by equilibrium equations. The stress resultants are related to the stresses by section integration. The stresses are related to strains by the material law. Finally, the strains are related to global deformations by kinematic equations. With the exception of statically determinate structures, in which equilibrium is sufficient to determine the internal forces, all structural analysis must include all these equations, plus boundary conditions.

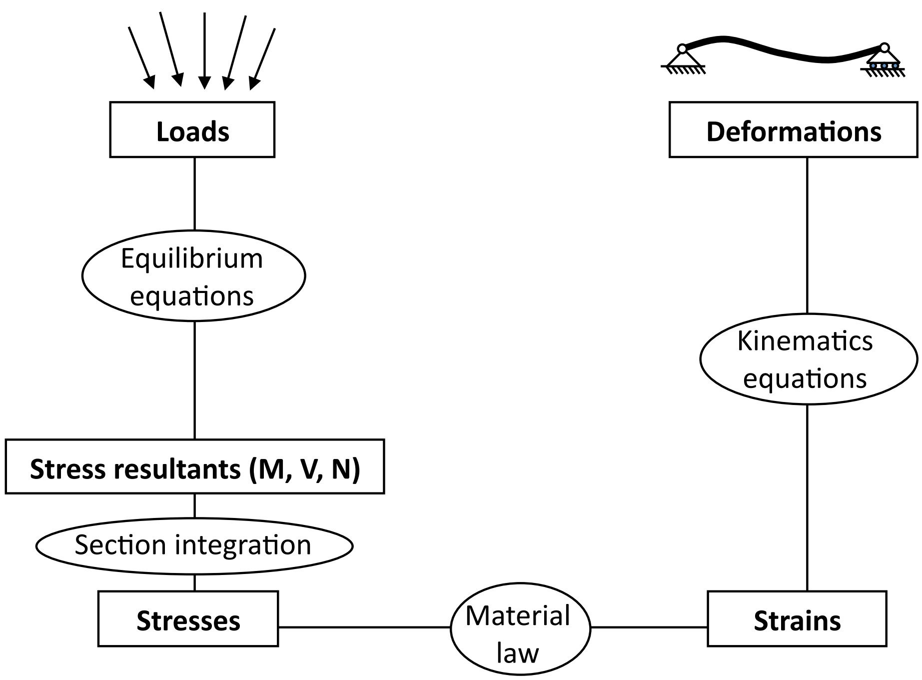

Figure 1: Ingredients of the structural analysis problem.

In the context of mathematics, the structural analysis problem is a boundary value problem (BVP). The governing differential equation is established by combining equilibrium, section integration, material law, and kinematic equations. In turn, solutions are obtained for particular problems by introducing boundary conditions and solving the differential equation. Additional solution approaches are available that are easier and more powerful than direct solution of the differential equation. These are listed at the end of this document.

Theories

Even within the theory of linear elastic analysis there exist a variety of theories that are named either according to its key assumption(s) or according to the name of the person who first formulated the theory. The following list may provide help with distinguishing between the various theories:

- **Theory of elasticity**
  - The material is assumed to be homogeneous and elastic
- **Linear elastic theory**
  - The popular special case of the theory of elasticity, in which the relationships between stresses and strains are proportional, i.e., Hooke's law. Also, all the governing equations are formulated with respect to the un-deformed geometry of the structure.
- **Nonlinear theory**
  - This phrase means that the material law or the kinematic equations have nonlinear terms
- **Material nonlinearity**
  - The material is assumed to be homogeneous, but without proportionality between stresses and strains
- **Geometric nonlinearity**
  - This phrase implies that large deformations are taken into account. In linear analysis the equilibrium equations are formulated in the un-deformed configuration of the structure. Conversely, geometric nonlinearity implies that the governing equations are updated at the deformed configuration, which necessitates an iterative algorithm to achieve equilibrium. The "snap-through" truss is the archtypical example of a geometrically nonlinear problem.
- **Second-order theory = Linearized second-order theory**
  - Buckling loads and P-Delta effects are calculated by this theory. This does not require iterations to achieve equilibrium like in the more general theory of geometric nonlinearity mentioned above. The phrase linearized applies because higher-order terms in the geometry of the displaced structure are cancelled.
- **P-Delta theory**
  - Same as the linearized second-order theory mentioned above
- **Small and large deformation theory**
  - Small deformations imply linear elasticity, while large deformations imply the geometrically nonlinear theory that is mentioned above.
- **Co-rotational theory**
  - A theory for geometrically nonlinear problems, i.e., problems in which account of large deformations is important.
- **Theory of plasticity**

- This is a theory of material nonlinearity, in which the material is assumed to have a specific yield stress

- **Elasto-plastic theory**
  - This is a special case of the theory of plasticity, in which the material is assumed to have zero stiffness after yielding. The assumption of elasto-plastic material response leads to two popular theorems: one for the lower bound of cross-section capacity and one for the upper bound of global structural load-carrying capacity.
- **Euler-Bernoulli beam theory**
  - In this theory the assumptions are made that the material is linearly elastic, according to Hooke's law, and that plane sections remain plane and perpendicular to the neutral axis during bending, according to Navier's hypothesis.
- **Timoshenko beam theory**
  - This theory adds shear deformation to the Euler-Bernoulli beam theory. This is done in the kinematic equations. The cross-sections are still assumed to remain plane, but no longer perpendicular to the neutral axis.
- **Kirchhoff and Kirchhoff-Love theory**
  - This is the theory for thin plates, i.e., plates in which shear deformation is negligible. This theory is analogue to the Euler-Bernoulli theory for beams.
- **Mindlin and Reissner-Mindlin theory**
  - This is the theory for thick plates, i.e., plates in which shear deformation is considerable. This theory is analogue to the Timoshenko beam theory.

Methods

In advanced structural analysis, one method dominates. It is called the finite element method, and it is built on the stiffness method. However, it is important to master several other structural analysis methods for hand calculation and to build understanding of structural behaviour. The objective of a structural method is to calculate the internal forces in the structure and/or to compute deformations. The map in Figure 2 provides an overview, and each method is described in a separate document. Note that for members like plates and shells the range of methods is limited to solution of the differential equation for simple problems or the full-blown finite element method. The arsenal of methods is greater for truss and frame structures, as shown in Figure 2.

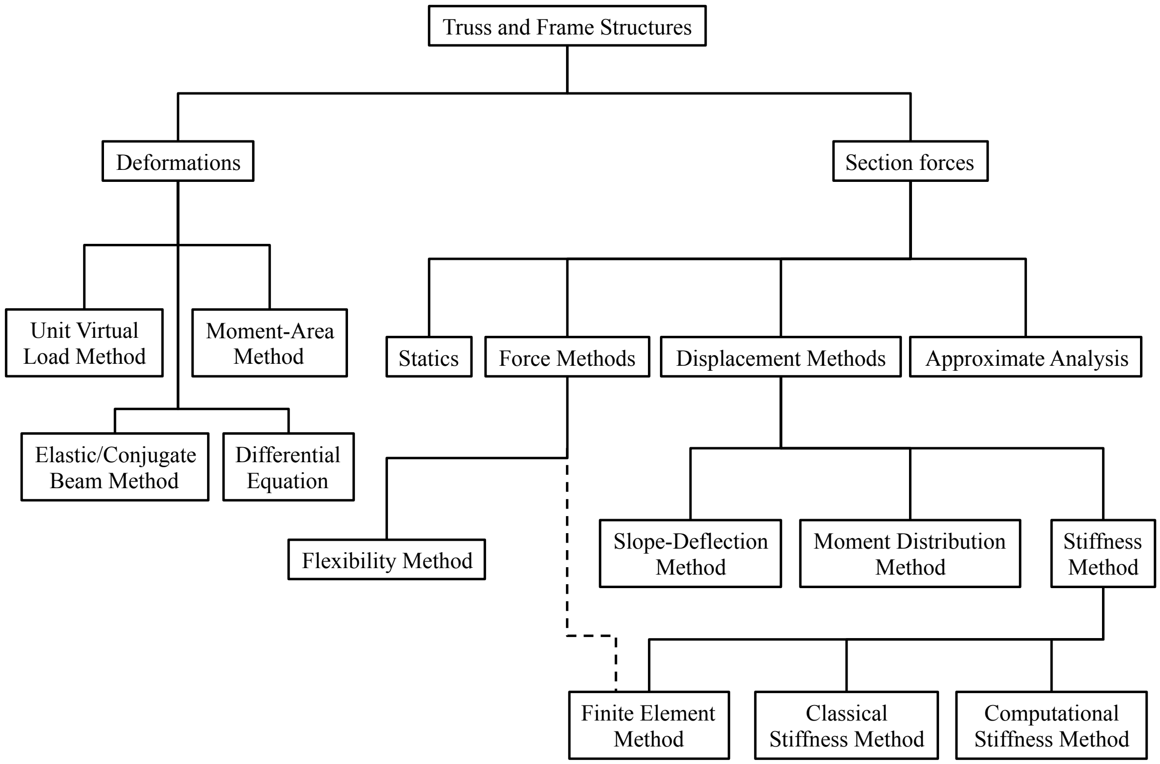

Figure 2: Overview of structural analysis methods.