Analysis of Functions

The starting point for reliability analysis is a systematic treatment of functions of random variables. Most engineering models and all limit-state functions are functions of random variables. In that context, this document forms the starting point for constructing reliability methods. The primary objective in this document is to determine the probability distribution or second-moment information of some dependent variables that are functions of a set of independent random variables. Another document on probability transformations deal with the problem of determining the functional relationship when it is the target probability distribution, or the second-moment information for the target random variables, that is known.

Second-moment Analysis

First, consider one function, Y, which is a function, $h(\mathbf{X})$, of n random variables, $\mathbf{X}$. Also assume for now that we have only second-moment information, i.e., mean, variance, and correlation, and that the second-moment information for Y is sought. To solve this problem, the expectation operator is vital. It is defined as

$$\mathrm{E}[Y] = \mathrm{E}[h(\mathbf{X})] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n \tag{1}$$

Expectation is a linear operator, having the properties

$$\begin{aligned} \mathrm{E}[a] &= a \\ \mathrm{E}[a \cdot h(\mathbf{X})] &= a \cdot \mathrm{E}[h(\mathbf{X})] \\ \mathrm{E}[h_1(\mathbf{X}) + h_2(\mathbf{X})] &= \mathrm{E}[h_1(\mathbf{X})] + \mathrm{E}[h_2(\mathbf{X})] \\ \frac{\partial}{\partial \theta} \mathrm{E}[h(\mathbf{X}, \theta)] &= \mathrm{E}[\frac{\partial}{\partial \theta} h(\mathbf{X}, \theta)] \end{aligned} \tag{2}$$

In passing it is noted that the variance, which is the expectation of $(X\text{-}\mu_X)^2$ has the properties

$$\begin{aligned} \mathrm{Var}[a + X] &= \mathrm{Var}[X] \\ \mathrm{Var}[b \cdot X] &= b^2 \cdot \mathrm{Var}[X] \end{aligned} \tag{3}$$

Some of these properties are useful to determine the mean and variance of Y. Consider first a linear function, where Y=$h(\mathbf{X})$ is written, in vector and index notation:

$$Y = a + \mathbf{b}^T \mathbf{X} = a + b_i X_i \tag{4}$$

where a is a constant and $\mathbf{b}$ is a vector of constants. The expectation of Eq. (4) is conveniently carried out in index notation:

$$\mu_Y = \mathrm{E}[Y] = a + b_i \cdot \mathrm{E}[X_i] = a + \mathbf{b}^T \mathbf{M}_X \tag{5}$$

Essentially, the mean of the function is obtained by substituting the means for the random variables.

The variance of the linear function in Eq. (4) is also derived in index notation:

$$\begin{aligned}
\sigma_Y^2 &= \mathrm{E}\left[\left(Y-\mu_Y\right)^2\right] \\
&= \mathrm{E}\left[\left(\left(a+b_iX_i\right)-\left(a+b_j\mu_j\right)\right)^2\right] \\
&= \mathrm{E}\left[\left(b_iX_i-b_j\mu_j\right)^2\right] \\
&= \mathrm{E}\left[\left(b_iX_i-b_j\mu_j\right)\cdot\left(b_kX_k-b_l\mu_l\right)\right] \\
&= \mathrm{E}\left[b_iX_i\cdot b_kX_k-b_iX_i\cdot b_l\mu_l-b_j\mu_j\cdot b_kX_k+b_j\mu_j\cdot b_l\mu_l\right] \\
&= \mathrm{E}\left[b_iX_i\cdot b_kX_k\right]-\mathrm{E}\left[b_iX_i\cdot b_l\mu_l\right]-\mathrm{E}\left[b_j\mu_j\cdot b_kX_k\right]+\mathrm{E}\left[b_j\mu_j\cdot b_l\mu_l\right] \\
&= b_ib_k\cdot\mathrm{E}\left[X_iX_k\right]-b_jb_l\cdot\mu_j\mu_l \\
&= b_ib_k\cdot\mathrm{Cov}\left[X_iX_k\right] \\
&= \mathbf{b}^T\mathbf{\Sigma}_{XX}\mathbf{b}
\end{aligned} \tag{6}$$

Similarly, one can also find the covariance between two different linear functions, $Y_1=a+\mathbf{b}^T\mathbf{X}$ and $Y_2=c+\mathbf{d}^T\mathbf{X}$:

$$\mathrm{Cov}\left[Y_1,Y_2\right]=\mathrm{E}\left[\left(Y_1-\mu_{Y_1}\right)\left(Y_2-\mu_{Y_2}\right)\right]=\mathbf{b}^T\mathbf{\Sigma}_{XX}\mathbf{d} \tag{7}$$

One may attempt to derive exact analytical second-moment expressions also for non-linear functions. However, depending on the complexity of the function, this may not be possible. In that case, one option is to approximate the function(s) by a Taylor expansion about the mean. Keeping the first two terms of the expansion yields the linearized approximation

$$Y=h(\mathbf{X})\approx h(\mathbf{M}_X)+\nabla h(\mathbf{M}_X)^T(\mathbf{X}-\mathbf{M}_X) \tag{8}$$

where, from mathematics, we know that $\nabla h(\mathbf{M}_X)$ is the gradient vector of the function, evaluated at the mean. According to the earlier derivations, the linearization in Eq. (8) yields the following second-moment results:

$$\mu_Y=h(\mathbf{M}_X) \tag{9}$$

$$\sigma_Y^2=\nabla h(\mathbf{M}_X)^T\mathbf{\Sigma}_{XX}\nabla h(\mathbf{M}_X) \tag{10}$$

$$\mathrm{Cov}[Y_1,Y_2]=\nabla h_1(\mathbf{M}_X)^T\mathbf{\Sigma}_{XX}\nabla h_2(\mathbf{M}_X) \tag{11}$$

Distributions

Reconsider the situation in which a dependent random variable, Y, is related by a known functional relationship to one or more independent random variables, $\mathbf{X}$. In this section the entire probability distribution, not only second-moment information of Y is sought.

This is often referred to as computing the "derived distribution." First consider the case of one random variable related to another: Suppose X and Y are related by the known functional relationship:

$$y = h(x) \quad \Leftrightarrow \quad x = h^{-1}(y) \tag{12}$$

where h is a monotonically increasing function so that there exists a one-to-one mapping between realizations x and y. The probability distribution for X is known, while the distribution for Y is sought. In general, the so-called probability preserving equation is applied to obtain the sought CDF:

$$F_Y(y) = F_X\left(h^{-1}(y)\right) \tag{13}$$

In turn, the CDF is differentiated to obtain the PDF. However, for one-to-one mappings it is possible to address the PDF directly:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X\left(h^{-1}(y)\right)}{dy} = \frac{dx}{dy} \cdot \frac{dF_X\left(h^{-1}(y)\right)}{dx} = \frac{dx}{dy} \cdot f_X(h^{-1}(y)) = \frac{dx}{dy} \cdot f_X(x) \tag{14}$$

Rearranging yields

$$f_Y(y) \cdot dy = f_X(x) \cdot dx \tag{15}$$

This equality is visualized by shaded areas in Figure 1.

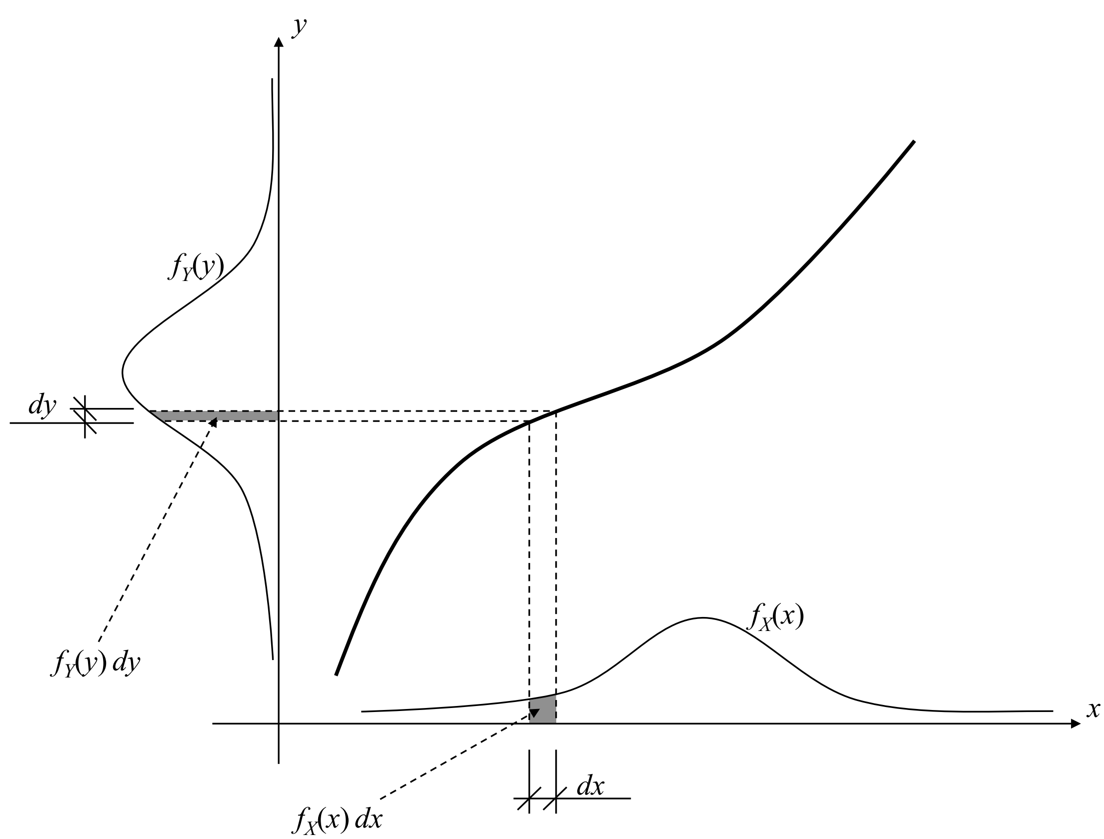

Figure 1: Derived distribution.

In the multi-variate case, depending on the distributions of the independent random variables, **X**, and the functional relationship with *Y* there may not exist an analytical expression for the distribution of *Y*. However, the following equality holds:

$$f_{\mathbf{Y}}(y_1, y_2, \cdots, y_n) dy_1 dy_2 \cdots dy_n = f_{\mathbf{X}}(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n \tag{16}$$

which by rearranging is written in terms of the Jacobian determinant:

$$f_{\mathbf{Y}}(y_1, y_2, \cdots, y_n) = f_{\mathbf{X}}(x_1, x_2, \cdots, x_n) \left| \det(\mathbf{J}_{\mathbf{y},\mathbf{x}}) \right|^{-1} \tag{17}$$

Note, however, that certain special cases are available: A linear function of normal random variables is always normal. A product function of lognormal random variables is always lognormal.